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The Eigenfunctions of Laplace's Tidal Equations over a Sphere

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THE EIGENFUNCTIONS OF LAPLACE'S TIDAL EQUATIONS OVER A SPHERE

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CONTENTS

Numerical calculations are presented for the eigenvalues of Laplace's tidal equations governing a thin layer of fluid on a rotating sphere, for a complete range of the parameter $\epsilon = 4\Omega^2R^2/gh$ (Ω = rate of rotation, R = radius, g = gravity, h = depth of fluid layer). The corresponding eigenfunctions or 'Hough functions' are shown graphically for the lower modes of oscillation. Negative values of ϵ, which have application in problems involving forced motions, are also considered.

The calculations reveal many asymptotic forms of the solution for various limiting values of ϵ. The corresponding analytical expressions are derived in the present paper.

Thus, as $\epsilon \to 0$ through positive values we have the well-known waves of the first and second class respectively, which were found by Margules and Hough. These can be represented in terms of spherical harmonics.

As $\epsilon \to +\infty$ there are three distinct asymptotic forms. In each of these the energy is concentrated near the equator. In the first type, the kinetic energy is three times the potential energy. In the other two types the kinetic and potential energies are equal. The waves of the second type are all propagated towards the west. The waves of the third type are Kelvin waves propagated eastwards along the equator. All three types are described in terms of Hermite polynomials.

As $\epsilon \to 0$ through negative values there is only one asymptotic form of solution, representing motions which are analytically continuous with Hough's 'waves of the second class'.

As $\epsilon \to -\infty$ there are three different asymptotic forms, in each of which the energy tends to be concentrated near the poles of rotation. In the first two types the energy is mainly kinetic and the motion is in inertial circles. In the third type the energy is mainly potential. The modes tend to occur in pairs of almost the same frequency, one being symmetric and the other antisymmetric about the equator. The analytical forms of the solutions involve generalized Laguerre polynomials.

In the special case of zonal oscillations, the first two limiting forms as $\epsilon \to -\infty$ go over into a different form in which the frequency tends to zero as ϵ tends to a finite negative value. In this case the third type does not occur.

The way in which the various asymptotic solutions are connected can be traced in figures 1 to 6 ($\epsilon > 0$) and figures 16 to 21 ($\epsilon < 0$). Accurate values of the eigenfrequencies, covering the range $-10^4 < \epsilon < 10^4$ are tabulated in tables 1 to 10. The eigenfunctions for the lower modes are presented graphically.

1. Introduction

This paper is concerned with the exact calculation of Laplace's tidal equations for a thin, uniform layer of fluid on the surface of a rotating sphere.

The problem is of fundamental importance for both the ocean and atmosphere (Wilkes 1949; Siebert 1961), yet until quite recently the computational effort required for its solution prevented a full numerical study. The solution depends upon a dimensionless parameter $\epsilon = 4\Omega^2 R^2/gh$ involving the rate of rotation Ω, the radius of the globe R, the acceleration of gravity g and the undisturbed depth of fluid h. At the time that this investigation was begun, only the limiting forms of the solutions as $\epsilon \to 0$ were adequately known. Thus Margules (1893) and Hough (1898) both pointed out that for small values of ϵ the solutions fall into two classes, namely the gravity waves (or waves of the first class) and the planetary waves (or waves of the second class). These authors each computed some representative values of the frequencies for both classes of waves, and Hough in particular derived some asymptotic expressions valid for small ϵ. On account of Hough's classical work, the functions describing the dependence of the pressure on the sine of the latitude have been called 'Hough functions'.

The mechanics of the waves of the second class (also called planetary waves) were discussed by Rossby (1939) who introduced the so-called 'β-plane' approximation, in which the surface of the sphere is locally approximated by a plane. Rattray (1964) and Rattray & Charnell (1966) have used a similar approximation to investigate solutions in the neighbourhood of the equator. The present author (1965) showed that for moderately large value of ϵ the solutions over the sphere were approximated by spheroidal wave functions. Some further calculations of the Hough functions for particular parameters corresponding to the Earth's atmosphere have been given by Haurwitz (1965). Here it may be mentioned that for the ordinary barotropic waves in the ocean and atmosphere the appropriate value of ϵ lies between 10 and 100, which is certainly not small, while for the baroclinic, or internal, waves, the appropriate value of ϵ is one or two orders of magnitude greater.

In certain special cases, for example the zonal solutions and those solutions having a frequency equal to 2Ω, the eigenfunctions are known to be expressible precisely in terms of spheroidal wavefunctions (see, for example, Eckart 1960). These particular cases, however, serve only to emphasize our ignorance of the form of the functions for general values of the frequency and longitudinal wavenumber.

The present investigation, begun in 1963, had the object of computing the eigenfunctions of Laplace's tidal equations over the complete range of values of ϵ. The main part of the computations were carried out by the method described in § 5. This provides an approximation converging rapidly for small values of ϵ. For large values of ϵ the method is still valid, though convergence is slower. The computations show that for sufficiently large, positive values of ϵ (that is, for high rates of rotation) the solutions are of three distinct types. In the first type the eigenfrequency is proportional asymptotically to $\epsilon^{-\frac{1}{4}}$, and in the other two types it is proportional to $\epsilon^{-\frac{1}{2}}$. The modes which for small ϵ are of the first *class* become of type 1 at large values of ϵ, with the exception of certain modes which are propagated eastwards along the equator and have the form of Kelvin waves; these are of type 3. Again the modes which for small ϵ are of the second *class* become of the type 2, with certain exceptions which become of type 1. In all cases as ϵ tends to infinity the energy tends to become more concentrated near the equator.

Similar results, but not including the waves of type 3, have been found by Golitsyn & Dikii (1966)†.

Analytical expressions for the asymptotic forms of the solution as $\epsilon \to \infty$ are derived in § 8. One curious feature of the waves of type 1 is that their kinetic energy is equal to three times their gravitational energy.

A further interesting discovery, also revealed by the computations, is the existence of periodic solutions corresponding to negative values of the parameter ϵ. These do not represent *free* modes of oscillation (unless one admits the oscillations of unstably stratified fluids). However, as Lindzen (1966) has pointed out, the eigenfunctions are useful in the representation of the response of the fluid system to applied forces, whether gravitational or thermal. An account of these solutions and of their asymptotic forms as $\epsilon \to -0$ or $\epsilon \to -\infty$ is given in §§ 10 to 13. It will appear that, as the rate of rotation is increased ($\epsilon \to -\infty$) the energy of these motions becomes trapped near the poles of rotation. Moreover, the eastward-going modes are found to exist only when ϵ exceeds a certain critical value, dependent on the particular mode.

In §§ 6, 9, 10 and 12 the numerical results are presented in the form of graphs which will give an idea of the form of the eigenfunctions for any particular value of ϵ, and of tables from which it is possible to extract the values of the eigenfrequencies. The conclusions are summarized in § 14.

2. Laplace's tidal equations

Imagine a thin layer of fluid of depth h on the surface of a gravitating solid sphere of unit radius, which rotates with angular velocity Ω. Let θ, ϕ represent the colatitude and longitude; t the time; u, v the eastward and northward components of velocity relative to the surface of the sphere; ζ the vertical displacement of the free surface from equilibrium level; and g the acceleration of gravity (assumed constant). Then Laplace's tidal equations may be written:

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(g\zeta) = 0, \tag{2·1}$$

$$\frac{\partial v}{\partial t} + 2\Omega u \sin\theta - \frac{\partial}{\partial\theta}(g\zeta) = 0, \tag{2·2}$$

$$\frac{\partial\zeta}{\partial t} + \frac{h}{\sin\theta}\left[\frac{\partial}{\partial\theta}(-v\sin\theta) + \frac{\partial u}{\partial\phi}\right] = 0. \tag{2·3}$$

† These results, with those described earlier were presented to the I.U.T.A.M. Symposium on Rotating Fluid Systems at La Jolla in March 1966. For an account of the Symposium see the report by Bretherton, Carrier & Longuet-Higgins (1966).

The validity of these equations for the ocean and atmosphere has been discussed elsewhere (for example, Hough 1898; Eckart 1960).

On multiplying equations (2·1), (2·2) and (2·3) by ρhu, ρhv and $\rho g\zeta$ respectively and adding we find

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\rho h(u^2+v^2)+\tfrac{1}{2}\rho g\zeta^2] = \rho h\left(u\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}\right)+\rho g\zeta\frac{\partial \zeta}{\partial t}$$

$$= \rho gh\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(u\zeta)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\zeta\sin\theta)\right]. \tag{2·4}$$

Now integrating over the surface of the sphere we obtain

$$\frac{\partial}{\partial t}\iint[\tfrac{1}{2}\rho h(u^2+v^2)+\tfrac{1}{2}\rho g\zeta^2]\,\mathrm{d}S = 0. \tag{2·5}$$

Thus the sum of the expressions

$$\left.\begin{aligned} I_1 &= \iint\tfrac{1}{2}\rho h(u^2+v^2)\,\mathrm{d}S,\\ I_2 &= \iint\tfrac{1}{2}\rho g\zeta^2\,\mathrm{d}S, \end{aligned}\right\} \tag{2·6}$$

is a constant. I_1 and I_2 will be called the kinetic and potential energies respectively.

We shall generally seek periodic solutions proportional to $e^{i(s\phi-\sigma t)}$, where s is a non-negative integer and σ a constant non-zero frequency. Then in equations (2·1) to (2·3) $\partial/\partial\phi$ is replaced by is and $\partial/\partial t$ by $-i\sigma$. Solving the first two equations for u and v now gives

$$\left.\begin{aligned} u &= \frac{1}{2\Omega(\lambda^2-\cos^2\theta)}\left(\frac{\lambda s}{\sin\theta}-\cos\theta\frac{\partial}{\partial\theta}\right)g\zeta,\\ v &= \frac{-i}{2\Omega(\lambda^2-\cos^2\theta)}\left(s\cot\theta-\lambda\frac{\partial}{\partial\theta}\right)g\zeta, \end{aligned}\right\} \tag{2·7}$$

where
$$\lambda = \sigma/2\Omega, \tag{2·8}$$

a non-dimensional frequency. On substituting for u and v in equation (2·3) we have

$$\mathscr{L}(\zeta) = \epsilon\zeta, \tag{2·9}$$

where $\mathscr{L}$ denotes the linear operator

$$\mathscr{L} \equiv \frac{1}{\lambda\sin\theta}\left[\frac{\partial}{\partial\theta}\left\{\frac{1}{\lambda^2-\cos^2\theta}\left(s\cos\theta-\lambda\sin\theta\frac{\partial}{\partial\theta}\right)\right\}+\frac{s}{\lambda^2-\cos^2\theta}\left(\frac{\lambda s}{\sin\theta}-\cos\theta\frac{\partial}{\partial\theta}\right)\right], \tag{2·10}$$

and where
$$\epsilon = 4\Omega^2/gh \tag{2·11}$$

(sometimes called Lamb's parameter). The problem then reduces to finding functions ζ, finite and with continuous derivatives in $0 \leqslant \theta \leqslant \pi$, and also pairs of constants λ and ϵ so as to satisfy equation (2·9).

In order to illustrate the significance of the variables we have purposely chosen a simple physical situation. However, as Taylor (1936) was the first to show, similar equations apply also to the free motions of a more general system in which the fluid is both stratified and compressible. In that case the variables u, v and ζ (or the pressure p) are functions also of the vertical coordinate, and in place of ϵ one has more generally

$$\epsilon' = 4\Omega^2/gh', \tag{2·12}$$

where h' is an 'equivalent depth'. Only in the case of barotropic motions does h' become nearly equal to the depth h.

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In the following we shall be concerned with the solution of the above equations for the functions u, v, ζ and the parameters ϵ, λ as a mathematical problem. As we shall see, there is more than one method of approach, each of which has its advantages. We begin with the method which is most powerful for the purpose of computation, and which yields most conveniently the asymptotic solutions as $\epsilon \to 0$.

3. First method of solution

Following Love (1913) we introduce functions Φ and Ψ, analogous to velocity potential and stream function, such that

$$\left.\begin{aligned} u &= \frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi} - \frac{\partial\Psi}{\partial\theta}, \\ v &= -\frac{\partial\Phi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\phi}, \end{aligned}\right\} \tag{3·1}$$

then

$$\left.\begin{aligned} \frac{1}{\sin\theta}\left[\frac{\partial u}{\partial\phi} - \frac{\partial}{\partial\theta}(v\sin\theta)\right] &= \nabla^2\Phi, \\ \frac{1}{\sin\theta}\left[-\frac{\partial v}{\partial\phi} - \frac{\partial}{\partial\theta}(u\sin\theta)\right] &= \nabla^2\Psi, \end{aligned}\right\} \tag{3·2}$$

where ∇^2 denotes the horizontal Laplacian operator,

$$\nabla^2 \equiv \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3·3}$$

The two expressions in (3·2) represent respectively the horizontal divergence and the vorticity.

Now let us operate on Laplace's equations as follows. Taking

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\phi}(2·1) - \frac{\partial}{\partial\theta}(\sin\theta(2·2))\right]$$

we find

$$\frac{\partial}{\partial t}\nabla^2\Phi + 2\Omega\cos\theta\,\nabla^2\Psi + 2\Omega\sin\theta\,u + g\nabla^2\zeta = 0, \tag{3·4}$$

and taking

$$\frac{1}{\sin\theta}\left[-\frac{\partial}{\partial\phi}(2·2) - \frac{\partial}{\partial\theta}(\sin\theta(2·1))\right],$$

we find

$$\frac{\partial}{\partial t}\nabla^2\Psi - 2\Omega\cos\theta\,\nabla^2\Phi - 2\Omega\sin\theta\,v = 0. \tag{3·5}$$

Equation (2·3) can also be written

$$\partial\zeta/\partial t + h\nabla^2\Phi = 0. \tag{3·6}$$

Substituting for u and v from equations (3·1) we obtain the following equations for Φ and Ψ:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\phi}\right)\Phi + 2\Omega\left(\cos\theta\,\nabla^2 - \sin\theta\frac{\partial}{\partial\theta}\right)\Psi &= -g\nabla^2\zeta, \\ \left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\phi}\right)\Psi - 2\Omega\left(\cos\theta\,\nabla^2 - \sin\theta\frac{\partial}{\partial\theta}\right)\Phi &= 0. \end{aligned}\right\} \tag{3·7}$$

We seek solutions to these equations which shall be proportional to $e^{i(s\phi-\sigma t)}$, where s is a non-negative integer and σ denotes the radian frequency. For convenience let us write

$$\sigma/2\Omega = \lambda, \qquad \cos\theta = \mu, \qquad (1-\mu^2)\frac{\partial}{\partial\mu} = \mathrm{D} \tag{3·8}$$

and define the non-dimensional parameter ϵ by

$$\epsilon = 4\Omega^2/gh \tag{3·9}$$

then equations (3·7) become

$$(\lambda\nabla^2 - s)\,\Phi + (\mu\nabla^2 + \mathrm{D})\,\mathrm{i}\Psi = -(\mathrm{i}g/2\Omega)\,\nabla^2\zeta, \tag{3·10}$$

$$(\lambda\nabla^2 - s)\,\mathrm{i}\Psi + (\mu\nabla^2 + \mathrm{D})\,\Phi = 0, \tag{3·11}$$

where now

$$\nabla^2 \equiv \frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}\right] - \frac{s^2}{1-\mu^2}. \tag{3·12}$$

Equation (3·6) becomes

$$\mathrm{i}\sigma\zeta = h\nabla^2\Phi. \tag{3·13}$$

On eliminating ζ from (3·10) and (3·13) we obtain

$$\left.\begin{aligned}\left(\lambda\nabla^2 - s + \frac{1}{\epsilon\lambda}\nabla^4\right)\Phi + (\mu\nabla^2 + \mathrm{D})\,\mathrm{i}\Psi = 0,\\ (\lambda\nabla^2 - s)\,\mathrm{i}\Psi + (\mu\nabla^2 + \mathrm{D})\,\Phi = 0.\end{aligned}\right\} \tag{3·14}$$

Let Φ and Ψ be expanded in series of spherical harmonics:

$$\left.\begin{aligned}\Phi &= \sum_{n=s}^{\infty} A_n^s P_n^s(\mu)\, e^{i(s\phi-\sigma t)},\\ \Psi &= \sum_{n=s}^{\infty} \mathrm{i}B_n^s P_n^s(\mu)\, e^{i(s\phi-\sigma t)}.\end{aligned}\right\} \tag{3·15}$$

Now we have

$$\nabla^2 P_n^s = -n(n+1)\,P_n^s, \tag{3·16}$$

and when $n > 0$

$$\left.\begin{aligned}\mu P_n^s &= \frac{n+s}{2n+1}P_{n-1}^s + \frac{n-s+1}{2n+1}P_{n+1}^s,\\ \mathrm{D}P_n^s &= \frac{(n+1)(n+s)}{2n+1}P_{n-1}^s - \frac{n(n-s+1)}{2n+1}P_{n+1}^s,\end{aligned}\right\} \tag{3·17}$$

so that

$$(\mu\nabla^2 + \mathrm{D})\,P_n^s = -\frac{(n-1)(n+1)(n+s)}{2n+1}P_{n-1}^s - \frac{n(n+2)(n-s+1)}{2n+1}P_{n+1}^s. \tag{3·18}$$

Substituting the series (3·15) into equation (3·14) and equating coefficients of P_n^s to zero we have

$$\left.\begin{aligned}&\left[-n(n+1)\,\lambda - s + \frac{n^2(n+1)^2}{\epsilon\lambda}\right]A_n^s\\ &\qquad + \left[\frac{n(n+2)(n+s+1)}{2n+3}B_{n+1}^s + \frac{(n-1)(n+1)(n-s)}{2n-1}B_{n-1}^s\right] = 0,\\ &[-n(n+1)\,\lambda - s]\,B_n^s - \left[\frac{n(n+2)(n+s+1)}{2n+3}A_{n+1}^s + \frac{(n-1)(n+1)(n-s)}{2n-1}A_{n-1}^s\right] = 0,\end{aligned}\right\} \tag{3·19}$$

that is to say
$$\left.\begin{aligned} K_n A_n^s + p_{n+1} B_{n+1}^s + q_{n-1} B_{n-1}^s = 0, \\ L_n B_n^s - p_{n+1} A_{n+1}^s - q_{n+1} A_{n-1}^s = 0, \end{aligned}\right\} \tag{3·20}$$

where
$$p_n = -\frac{(n+1)(n+s)}{n(2n+1)}, \qquad q_n = -\frac{n(n-s+1)}{(n+1)(2n+1)}, \tag{3·21}$$

$$K_n = \lambda + \frac{s}{n(n+1)} - \frac{n(n+1)}{\epsilon\lambda}, \qquad L_n = \lambda + \frac{s}{n(n+1)} \tag{3·22}$$

and $n = s, (s+1), (s+2), \ldots$. The equations fall into two independent systems as follows. We have
$$\begin{pmatrix} K_s & p_{s+1} & 0 & 0 & \ldots \\ q_s & L_{s+1} & p_{s+2} & 0 & \ldots \\ 0 & q_{s+1} & K_{s+2} & p_{s+3} & \ldots \\ 0 & 0 & q_{s+2} & L_{s+3} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} A_s^s \\ B_{s+1}^s \\ A_{s+2}^s \\ B_{s+3}^s \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{3·23}$$

and
$$\begin{pmatrix} L_s & p_{s+1} & 0 & 0 & \ldots \\ q_s & K_{s+1} & p_{s+2} & 0 & \ldots \\ 0 & q_{s+1} & L_{s+2} & p_{s+3} & \ldots \\ 0 & 0 & q_{s+2} & K_{s+3} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} B_s^s \\ A_{s+1}^s \\ B_{s+2}^s \\ A_{s+3}^s \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{3·24}$$

The two systems correspond to motions which are respectively symmetric and anti-symmetric about the equator. For the first system the motions are mirrored in the equatorial plane, and there is no motion across the equator. In the second system the motion at the equator is normal to the equator.

The matrix of the system (3·23) may be written
$$\lambda \mathbf{I} - \mathbf{C} - (1/\epsilon\lambda)\,\mathbf{J}, \tag{3·25}$$

where **I** denotes the unit matrix, **J** denotes the diagonal matrix
$$\mathbf{J} = \begin{pmatrix} s(s+1) & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & (s+2)(s+3) & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & (s+4)(s+5) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3·26}$$

of which every alternate diagonal element is zero, and **C** denotes the matrix
$$\mathbf{C} = \begin{pmatrix} \dfrac{-s}{s(s+1)} & \dfrac{(s+2)(2s+1)}{(s+1)(2s+3)} & 0 & \ldots \\ \dfrac{s\,.\,1}{(s+1)(2s+1)} & \dfrac{-s}{(s+1)(s+2)} & \dfrac{(s+3)(2s+2)}{(s+2)(2s+5)} & \ldots \\ 0 & \dfrac{(s+1)\,2}{(s+2)(2s+3)} & \dfrac{-s}{(s+2)(s+3)} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3·27}$$

in which the only non-vanishing elements are on the three central diagonals. These are given by

$$\left.\begin{aligned} C_{ii} &= \frac{-s}{(s+i-1)\,(s+i)}, \\ C_{i,i+1} &= \frac{(s+i+1)\,(2s+i)}{(s+i)\,(2s+2i+1)}, \\ C_{i+1,i} &= \frac{(s+i-1)\,i}{(s+i)\,(2s+2i-1)}. \end{aligned}\right\} \tag{3·28}$$

For the antisymmetric modes the matrix of the system is identical except that $\mathbf{J}$ must be replaced by

$$J' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & (s+1)\,(s+2) & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & (s+3)\,(s+4) & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3·29}$$

It may be pointed out that the above system of equations is somewhat simpler than the equivalent systems used by Hough (1898), Love (1913) or Dikii (1961, 1965).

Consider, for example, the system (3·23). In order that this shall have solutions, the determinant of the system must vanish. That is to say we must have

$$|\lambda\mathbf{I}-\mathbf{C}-(1/\epsilon\lambda)\,\mathbf{J}| = 0. \tag{3·30}$$

The solution of this equation for λ will give the frequencies of the normal modes.†

4. Asymptotic forms of the solutions as $\epsilon \to 0$

The nature of the solutions for small values of $\epsilon\lambda$ can be seen at once. For then $1/\epsilon\lambda$ is a large quantity and the equation reduces approximately to

$$\prod_{i=1}^{\infty}\left(\lambda - C_{ii} - \frac{1}{\epsilon\lambda}J_{ii}\right) \doteqdot 0. \tag{4·1}$$

The odd factors give

$$\lambda + \frac{s}{(s+i-1)\,(s+i)} - \frac{1}{\epsilon\lambda}(s+i-1)\,(s+i) \doteqdot 0, \tag{4·2}$$

that is
$$\lambda \doteqdot -\frac{s}{n(n+1)} \pm \sqrt{\left\{\frac{s^2}{4n^2(n+1)^2} + \frac{n(n+1)}{\epsilon}\right\}}, \tag{4·3}$$

where $n = s+i-1$, an integer. For large ϵ, we have

$$\lambda \doteqdot \pm\sqrt{\frac{n(n+1)}{\epsilon}}, \tag{4·4}$$

and so
$$\sigma = 2\Omega\lambda \doteqdot \sqrt{\{n(n+1)\,gh\}}. \tag{4·5}$$

These are the gravity waves, or waves of the first class. They are described very nearly by

† It should be noted that when $s = 0$ the leading term of (3·27), which has the indeterminate form 0/0, must be set equal to zero.

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a single spherical harmonic:
$$\left.\begin{aligned}\Phi &\doteqdot A_n^s P_n^s(\mu)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)},\\ \Psi &\doteqdot 0.\end{aligned}\right\} \tag{4·6}$$

Higher terms in this approximation have been given by Hough (1898), Blinova (1960) and Dikii (1961, 1966).

On the other hand, the even factors in (3·28) give
$$\lambda+\frac{s}{(s+i-1)\,(s+i)} \doteqdot 0, \tag{4·7}$$
and so
$$\lambda \doteqdot -\frac{s}{n(n+1)}, \tag{4·8}$$
or
$$\sigma \doteqdot -\frac{2\Omega s}{n(n+1)}. \tag{4·9}$$
The solutions are described very nearly by
$$\left.\begin{aligned}\Phi &\doteqdot 0,\\ \Psi &\doteqdot \mathrm{i}B_n^s P_n^s(\mu)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}.\end{aligned}\right\} \tag{4·10}$$

These are the solutions of the second class, or the planetary waves (Rossby waves). Their frequency is proportional nearly to Ω, the fundamental rate of rotation. Higher terms in the expansion of σ in powers of ϵ have been given by Hough (1898) and Dikii (1961, 1966).

5. Method of computation for general values of ϵ

Let us return to the solution of equation (3·30) in the general case. (3·30) is an eigenvalue equation, but not of the usual kind. For λ occurs both as multiplying the matrix $\mathbf{I}$ and as dividing $\mathbf{J}/\epsilon$. We may proceed as follows. Define a new parameter
$$\eta = 1/\epsilon\lambda, \tag{5·1}$$
and let
$$\mathbf{C}+\eta\mathbf{J} = \mathbf{D}, \tag{5·2}$$
say. Then equation (3·30) becomes
$$|\lambda\mathbf{I}-\mathbf{D}| = 0, \tag{5·3}$$
which is an ordinary eigenvalue equation. Having found a sequence of eigenvalues λ_i one may then determine the corresponding values of ϵ by the relation
$$\epsilon_i = 1/\eta\lambda_i. \tag{5·4}$$

This gives us a sequence of pairs of values of ϵ, λ which may be sufficient for plotting λ as a (many-valued) function of ϵ. Then, if we wish we may find λ for any given value of ϵ by interpolation and successive approximation.

It is convenient for the purpose of computation to replace the unsymmetrical determinant C by a symmetric determinant as follows. Multiply the ith row by α_i and the ith column by α_i^{-1} where†
$$\left.\begin{aligned}\alpha_1 &= 1,\\ \alpha_i &= \left(\frac{C_{12}\,C_{23}\ldots C_{i-1,i}}{C_{21}\,C_{32}\ldots C_{i,i-1}}\right)^{\frac{1}{2}} \quad (i \geqslant 2).\end{aligned}\right\} \tag{5·5}$$

† In the case $s = 0$, C_{21} vanishes so that the first row and column must be omitted from the symmetrization. The process can still be carried out on the remaining rows and columns. The first element in the eigenvector can then be computed from the other elements.

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This will not affect the value of the determinant, or any of the diagonal elements. On the other hand, the matrix **C** will be replaced by the matrix **C*** where

$$C^*_{i,i+1} = C_{i,i+1}\alpha_i\alpha_{i+1}^{-1} = (C_{i,i+1}C_{i+1,i})^{\frac{1}{2}} \tag{5·6}$$

and similarly for $C^*_{i+1,i}$. The other elements of **C*** vanish. Thus **C*** is symmetric.

In practice the matrix $(\lambda\mathbf{I}-\mathbf{C}^*)$ must be truncated after a finite number of rows and columns. Let this number be N. The calculation will then yield N eigenvalues (distinct or otherwise). Successive approximations may be obtained by increasing N until a sufficient number of eigenvalues have converged to the desired degree of approximation.

Among the eigenvalues for any particular value of η we may expect some to be positive and some to be negative. Those eigenvalues λ which are of the same sign as $\eta = 1/\epsilon\lambda$, will correspond to positive values of ϵ, that is to say to positive depths h. These will be discussed first. The eigenvalues corresponding to negative values of ϵ will be discussed in §§ 9 to 12.

6. The eigenvalues for $\epsilon > 0$

As a first step, the eigenvalues λ were computed for the following values of η:

$$\eta = \pm\surd 2^k \quad (k = 24, 23, \ldots, -24). \tag{6·1}$$

When $|\eta| > 1$ it was found that sufficient accuracy in the lower modes was obtained by taking $N = 30$. To be more precise, when $|\eta| > 1$ the values five lowest positive symmetric or of the five lowest positive antisymmetric modes when $N = 30$ differed from the corresponding values when $N = 20$ by less than one part in 10^9. When $|\eta| \leqslant 1$, similar accuracy was obtained by taking $N = 50$, provided $\epsilon = 1/\lambda\eta$ was not less than 0·02. For negative values of ϵ, similar accuracy was obtained except for some instances in the waves of the second class. In all cases the errors accepted were less than one part in 10^4.

The eigenvalues found in this way have been tabulated in tables 2 and 3 (corresponding to $s = 1$ to 5) and table 1 (corresponding to $s = 0$). The entries in each case are believed correct to the number of decimal places given.

The entries in tables 1 to 3, together with a few further values, are the basis of the curves in figures 1 to 6. In these the eigenvalues have been plotted as a function of the parameter

$$\gamma = \epsilon^{-\frac{1}{2}} = \surd(gh)/2\Omega. \tag{6·2}$$

Several features are at once apparent. The eigenfrequencies are in every case monotonic functions of γ. As $\gamma \to \infty$ $(\epsilon \to 0)$ they approach the limiting values given by equations (4·4) and (4·6). The corresponding values of $(n-s)$ are shown against each curve.

On the other hand as $\gamma \to 0$ $(\epsilon \to \infty)$ we see that λ always tends to zero like some negative power of ϵ. (The symbols ν, ν' and ν'' correspond to the asymptotic forms derived in § 8.) To investigate the behaviour of the eigenfunctions in this part of the range of ϵ we must adopt a different approach as follows.

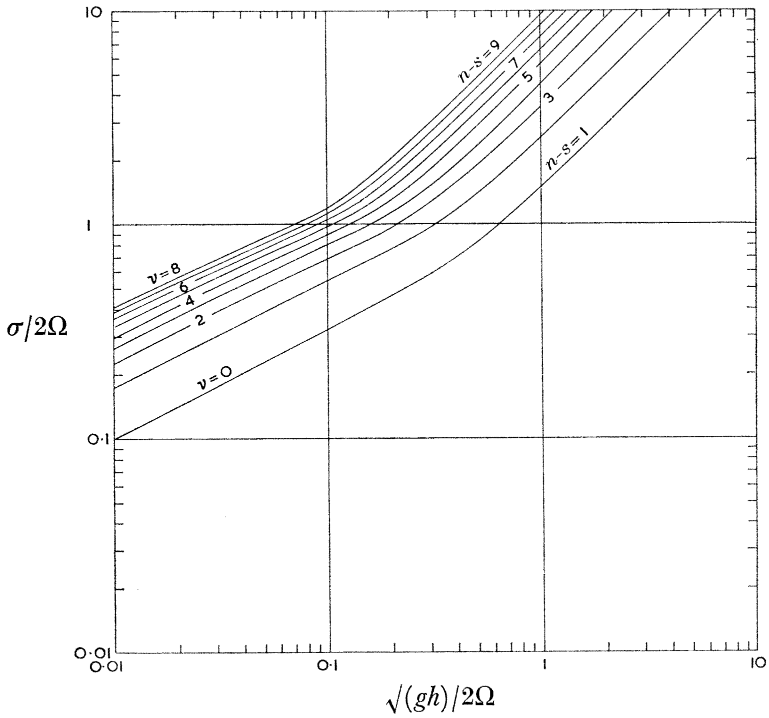

FIGURE 1. Eigenfrequencies of free modes of oscillation when $s = 0$.

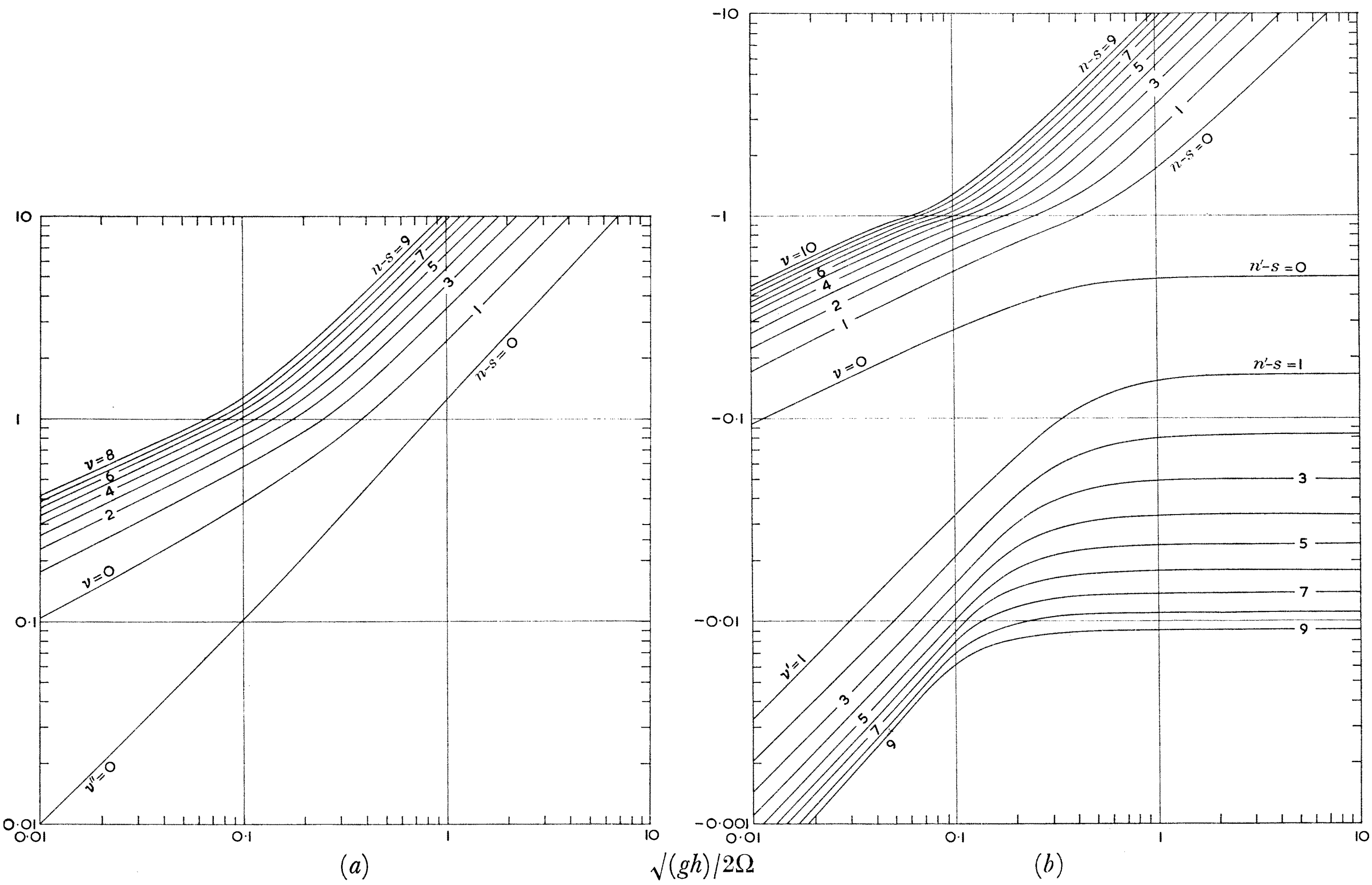

FIGURE 2. Eigenfrequencies of free modes of oscillation on the sphere when $s = 1$: (a) modes travelling eastwards, (b) modes travelling westwards.

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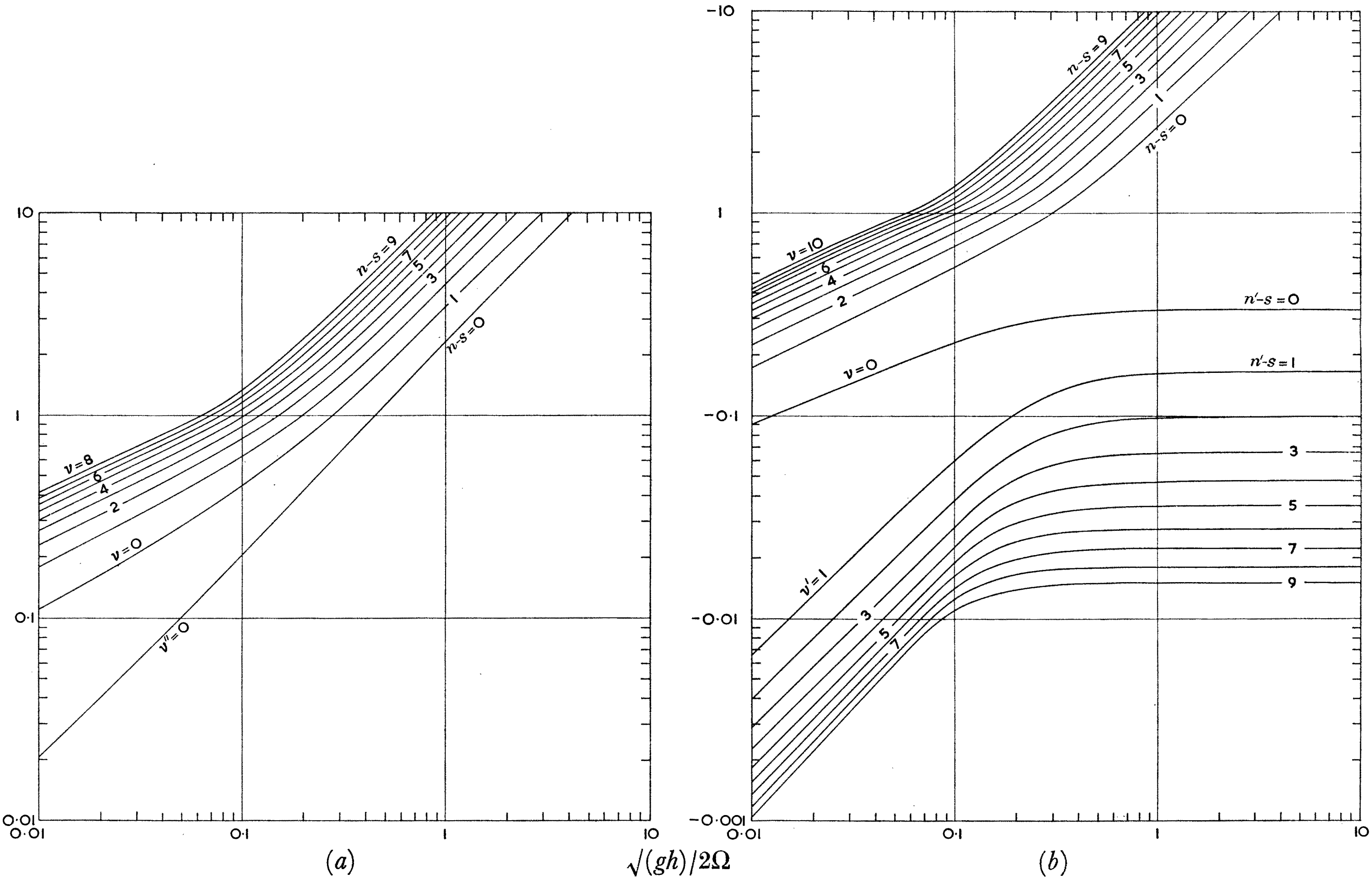

FIGURE 3. Eigenfrequencies of free modes of oscillation on the sphere when $s = 2$: (*a*) modes travelling eastwards, (*b*) modes travelling westwards.

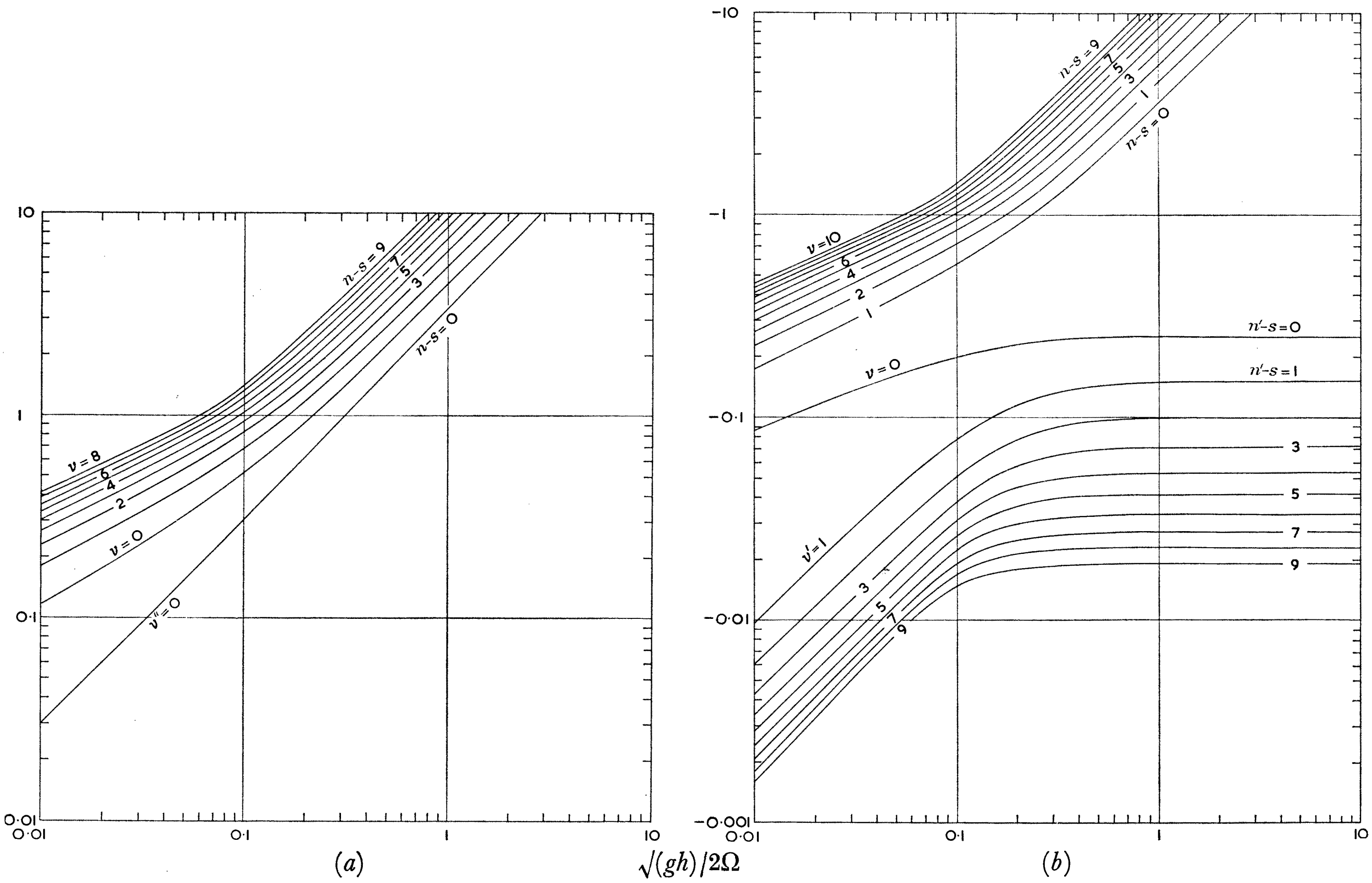

FIGURE 4. Eigenfrequencies of free modes of oscillation on the sphere when $s = 3$: (*a*) modes travelling eastwards, (*b*) modes travelling westwards.

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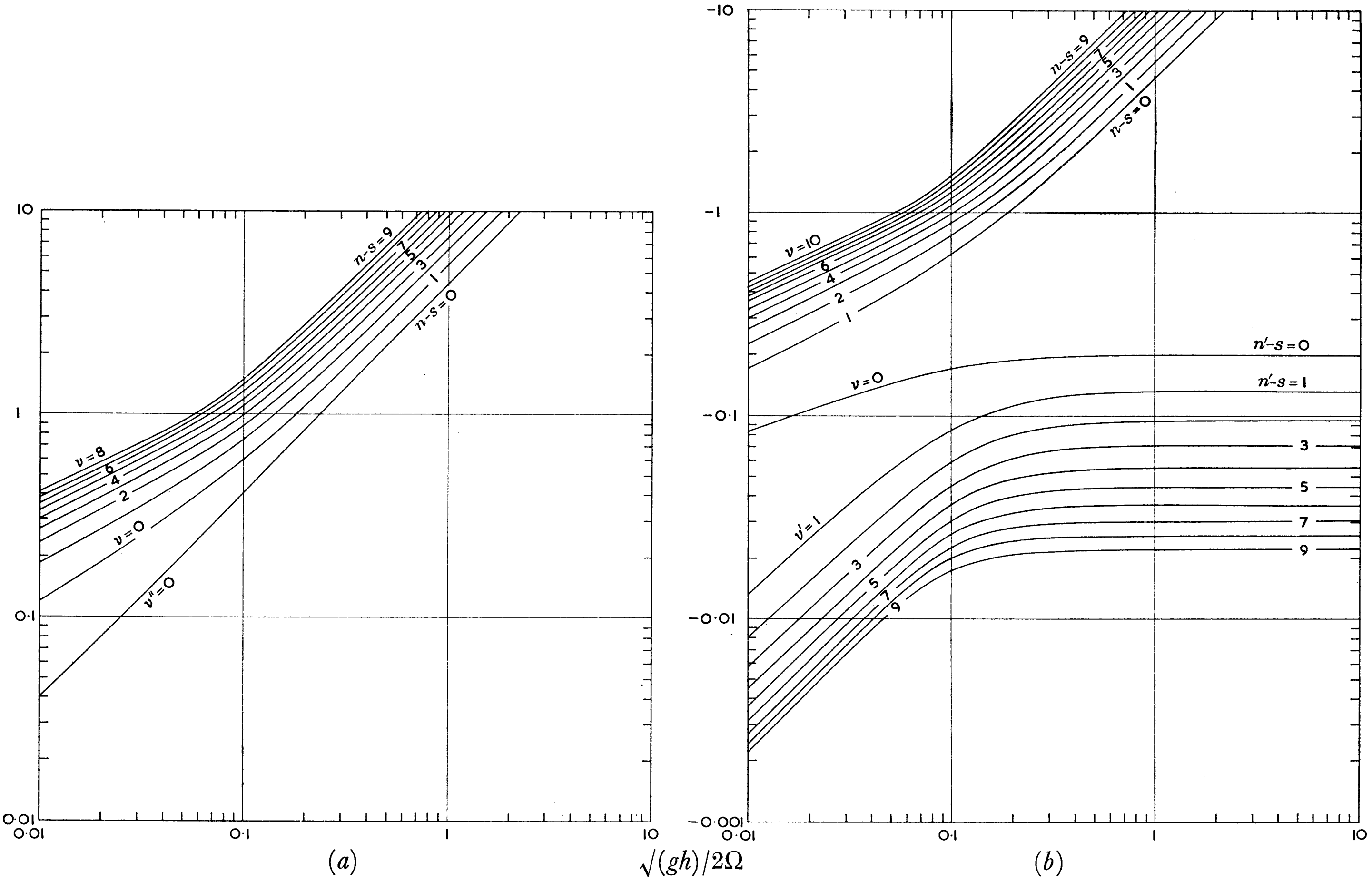

FIGURE 5. Eigenfrequencies of free modes of oscillation on the sphere when $s = 4$: (*a*) modes travelling eastwards, (*b*) modes travelling westwards.

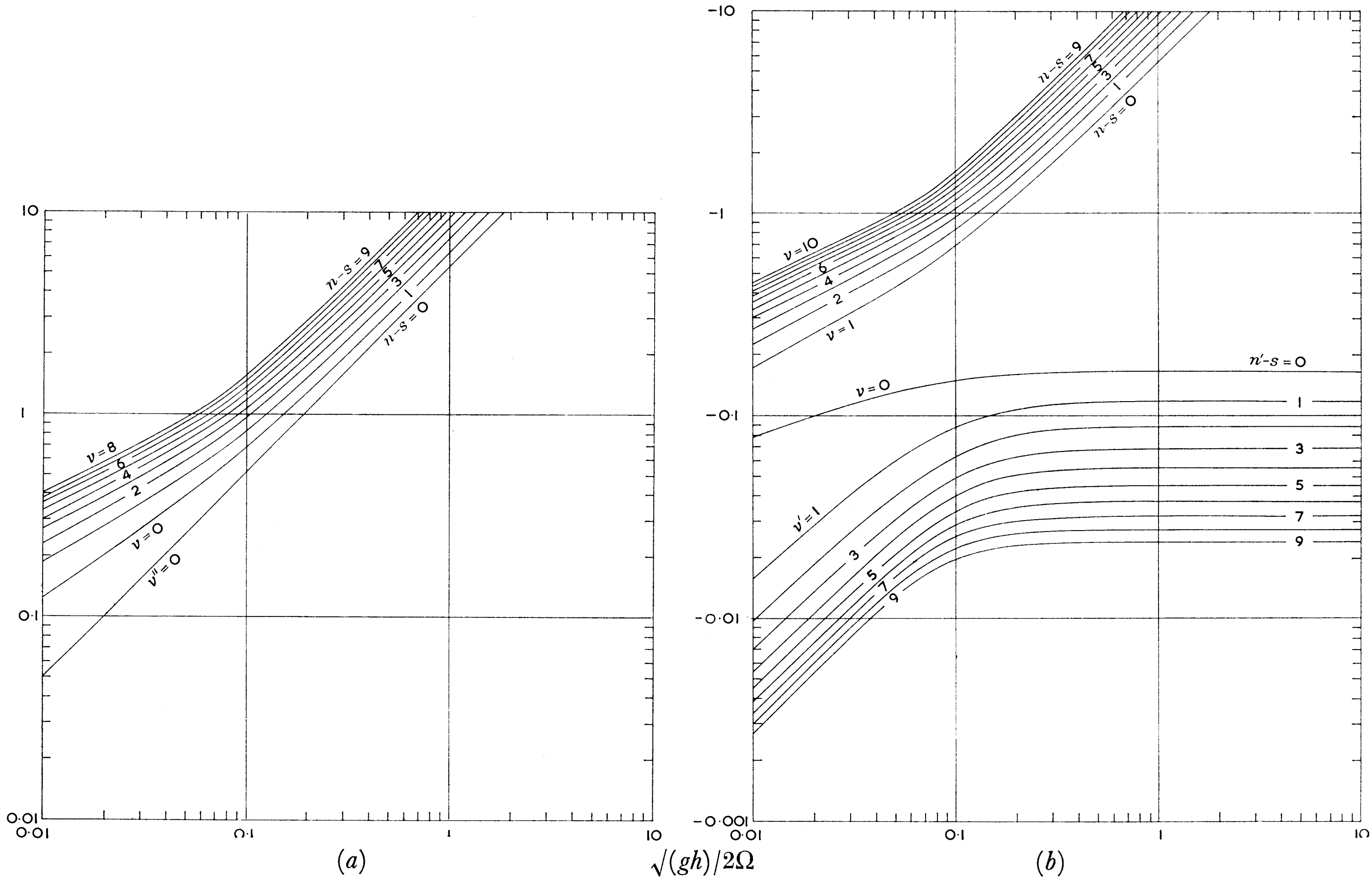

FIGURE 6. Eigenfrequencies of free modes of oscillation on the sphere when $s = 5$: (*a*) modes travelling eastwards, (*b*) modes travelling westwards.

7. Second method of solution

In Laplace's equations (§ 2) let us again seek periodic solutions proportional to $e^{i(s\phi-\sigma t)}$, taking as new dependent variables (cf. Margules 1893)

$$\left.\begin{aligned} u^* &= u\sin\theta, \\ v^* &= iv\sin\theta, \\ \zeta^* &= g\zeta/2\Omega. \end{aligned}\right\} \tag{7·1}$$

These equations then assume the simpler form

$$\left.\begin{aligned} \lambda u^* - \mu v^* - s\zeta^* &= 0, \\ \mu u^* - \lambda v^* + \mathrm{D}\zeta^* &= 0, \\ su^* - \mathrm{D}v^* - \epsilon\lambda(1-\mu^2)\,\zeta^* &= 0, \end{aligned}\right\} \tag{7·2}$$

where we have written

$$\lambda = \sigma/2\Omega, \quad \mu = \cos\theta, \quad \mathrm{D} = (1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}. \tag{7·3}$$

On eliminating u^* from the last two of equations (7·2) by means of the first, we have the simultaneous pair

$$\left.\begin{aligned} (\lambda\mathrm{D}+s\mu)\,\zeta^* &= (\lambda^2-\mu^2)\,v^*, \\ (\lambda\mathrm{D}-s\mu)\,v^* &= \{s^2-\epsilon\lambda^2(1-\mu^2)\}\,\zeta^*. \end{aligned}\right\} \tag{7·4}$$

Hence as equations for v^* and ζ^* we obtain

$$\left.\begin{aligned} \left[(\lambda\mathrm{D}+s\mu)\left\{\frac{1}{s^2-\epsilon\lambda^2(1-\mu^2)}(\lambda\mathrm{D}-s\mu)\right\}-(\lambda^2-\mu^2)\right]v^* &= 0, \\ \left[(\lambda\mathrm{D}-s\mu)\left\{\frac{1}{\lambda^2-\mu^2}(\lambda\mathrm{D}+s\mu)\right\}-\{s^2-\epsilon\lambda^2(1-\mu^2)\}\right]\zeta^* &= 0. \end{aligned}\right\} \tag{7·5}$$

Taking account of the identity

$$(\lambda\mathrm{D}+s\mu)\,(\lambda\mathrm{D}-s\mu) \equiv \lambda(1-\mu^2)\,(\lambda\nabla^2-s)+s^2(\lambda^2-\mu^2) \tag{7·6}$$

where

$$\nabla^2 \equiv \frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}\right]-\frac{s^2}{1-\mu^2}, \tag{7·7}$$

we have

$$\left[(\lambda\nabla^2-s)-\frac{2\epsilon\lambda^2\mu}{s^2-\epsilon\lambda^2(1-\mu^2)}(\lambda\mathrm{D}-s\mu)+\epsilon\lambda(\lambda^2-\mu^2)\right]v^* = 0, \tag{7·8}$$

and

$$\left[(\lambda\nabla^2+s)+\frac{2\mu}{\lambda^2-\mu^2}(\lambda\mathrm{D}+s\mu)+\epsilon\lambda(\lambda^2-\mu^2)\right]\zeta^* = 0. \tag{7·9}$$

Each of the equations has an apparent singularity where the expression in the denominator vanishes; but on examination the singularities turn out to be removable.

These equations enable us to determine the asymptotic forms of the eigenfunctions as $\lambda \to 0$ and as $\epsilon \to \pm\infty$.

8. Asymptotic forms of the solutions as $\epsilon \to \infty$

First let us proceed heuristically. If in equation (7·8) $\epsilon\lambda$ is large, then the last term $\epsilon\lambda(\lambda^2-\mu^2)\,v^*$ will tend to predominate over the first two. However, in order to satisfy the boundary conditions at both $\mu = \pm 1$, the first term $(\lambda\nabla^2-s)\,v^*$ which contains the highest derivative with respect to μ, must be retained. To make this first term comparable with the last the scale of variation of μ must be small. The change of scale will also make the first group of terms large compared to the second, which involves only the derivative $d/d\mu$. Hence we shall have a balance between the first and third groups of terms, that is to say

$$[(\lambda\nabla^2 - s) + \epsilon\lambda(\lambda^2 - \mu^2)]\, v^* = 0. \tag{8·1}$$

The precise order of magnitude of the terms neglected in this equation will be determined later.

Equation (8·1) may be written

$$(\nabla^2 + A - \epsilon\mu^2)\, v^* = 0, \tag{8·2}$$

where
$$A = -s/\lambda + \epsilon\lambda^2. \tag{8·3}$$

Equation (8·2) is the standard form of the spheroidal wave equation, which has been used as a basis of approximation in a previous paper (Longuet-Higgins 1965).†

Now when $\epsilon\lambda$ is large, equation (8·1) is most easily satisfied when $(\lambda^2 - \mu^2) \ll 1$. If λ also is small, then we expect $\mu^2 \ll 1$, that is to say the solution is confined to the neighbourhood of the equator. Now when μ is small, we see from (7·7) that

$$\nabla^2 \doteqdot d^2/d\mu^2. \tag{8·4}$$

This suggests the substitution
$$\eta = \epsilon^{\frac{1}{4}}\mu, \tag{8·5}$$

by which equation (8·2) becomes

$$\left(\frac{d^2}{d\eta^2} + \frac{A}{\epsilon^{\frac{1}{2}}} - \eta^2\right) v^* = 0, \tag{8·6}$$

a form of Weber's equation. In order for solutions to exist which are finite as $\eta \to \pm\infty$ we must have
$$A/\epsilon^{\frac{1}{2}} = 2\nu + 1 \qquad (\nu = 0, 1, 2, \ldots), \tag{8·7}$$

and then
$$v^* \propto e^{-\frac{1}{2}\eta^2} H_\nu(\eta), \tag{8·8}$$

H_ν being the Hermite polynomial of order ν. The function v^* is exponentially small beyond the turning-points
$$\eta = \pm\surd(2\nu + 1). \tag{8·9}$$

Hence the solution is indeed confined to the neighbourhood of the equator (unless ν is very large).

The eigenvalues are found from equation (8·3). Eliminating A from (8·3) and (8·7) we have
$$-s/\lambda + \epsilon\lambda^2 = (2\nu + 1)\,\epsilon^{\frac{1}{2}}. \tag{8·10}$$

This gives us a cubic equation for λ, namely

$$\lambda^3 - (2\nu + 1)\,\lambda/\epsilon^{\frac{1}{2}} - s/\epsilon = 0. \tag{8·11}$$

For large values of ϵ it is easily verified that the solutions of the cubic are given by

$$\lambda \doteqdot \pm\frac{(2\nu + 1)^{\frac{1}{2}}}{\epsilon^{\frac{1}{4}}} + \frac{s}{(4\nu + 2)\,\epsilon^{\frac{1}{2}}} \tag{8·12}$$

and
$$\lambda \doteqdot -\frac{s}{(2\nu + 1)\,\epsilon^{\frac{1}{2}}}. \tag{8·13}$$

The leading terms in these expansions are proportional to $\epsilon^{-\frac{1}{4}}$ and to $\epsilon^{-\frac{1}{2}}$ respectively.

† Equation (8·2) is also derived by Dikii (1966), who points out that the same equation occurs in quantum mechanics. From this point on Dikii follows a somewhat different line of argument.

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To verify the correctness of these expressions we may now return to the original equation (7·8) and make the substitution (8·5) with either $\lambda = \epsilon^{-\frac{1}{4}}L$ or $\lambda = \epsilon^{-\frac{1}{2}}L$, L being of order unity. It can thus be shown that the differential equation (8·6) and the limiting forms (8·8) to (8·13) are correct to the order stated, provided however that the denominator in (7·8) is not small over a significant range of μ. The exceptional case occurs when $\epsilon\lambda^2 \doteqdot s^2$ and $\mu \ll 1$. Hence in (8·13) the value $\nu = 0$ must be excluded. (However, in (8·12) $\nu = 0$ is permissible.)

To investigate the exceptional case we may try first the substitution

$$\lambda = \frac{s}{\epsilon^{\frac{1}{2}}} + \tfrac{1}{2}q\frac{s}{\epsilon} + O\left(\frac{1}{\epsilon^{\frac{3}{2}}}\right), \tag{8·14}$$

where q is a constant to be determined. Then we have

$$s^2 - \epsilon\lambda^2(1-\mu^2) = s^2[\mu^2 - q\epsilon^{-\frac{1}{2}}(1-\mu^2)] + O(\epsilon^{-1}). \tag{8·15}$$

If now in equation (6·8) we substitute

$$\eta = \epsilon^{\frac{1}{4}}\mu, \tag{8·16}$$

where η is of order unity, then that equation, after retaining only the terms of highest order, reduces to

$$\left[\left(\frac{d^2}{d\eta^2} - 1\right) - \frac{2\eta}{\eta^2 - q}\left(\frac{d}{d\eta} - \eta\right) - \eta^2\right]v^* = 0, \tag{8·17}$$

which may be rewritten in the form

$$\left(\frac{d}{d\eta} + \eta\right)\left[\frac{1}{\eta^2 - q}\left(\frac{d}{d\eta} - \eta\right)v^*\right] = 0, \tag{8·18}$$

that is to say

$$\left(\frac{d}{d\eta} + \eta\right)Z = 0, \tag{8·19}$$

where

$$Z = \frac{1}{\eta^2 - q}\left(\frac{d}{d\eta} - \eta\right)v^*. \tag{8·20}$$

The general solution of (8·19) is

$$Z = A\,e^{-\frac{1}{2}\eta^2}, \tag{8·21}$$

where A is an arbitrary constant. So on substitution in (8·20) we have

$$\left(\frac{d}{d\eta} - \eta\right)v^* = A(\eta^2 - q)\,e^{-\frac{1}{2}\eta^2} \tag{8·22}$$

of which the general solution is

$$v^* = A\,e^{\frac{1}{2}\eta^2}\int_0^\eta (\eta^2 - q)\,e^{-\eta^2}\,d\eta + B\,e^{\frac{1}{2}\eta^2}. \tag{8·23}$$

The term in $\eta^2\,e^{-\frac{1}{2}\eta^2}$ may be integrated by parts to give

$$v^* = -A\eta\,e^{-\frac{1}{2}\eta^2} + A(\tfrac{1}{2} - q)\,e^{\frac{1}{2}\eta^2}\int_0^\eta e^{-\eta^2}\,d\eta + B\,e^{\frac{1}{2}\eta^2}. \tag{8·24}$$

If v^* is to be finite at $\eta = \pm\infty$ we must have

$$q = \tfrac{1}{2}, \qquad B = 0. \tag{8·25}$$

Taking $A = -1$ we have then

$$v^* = \eta\,e^{-\frac{1}{2}\eta^2}. \tag{8·26}$$

From (8·14) the eigenvalue λ is given by

$$\lambda = \frac{s}{\epsilon^{\frac{1}{2}}}+\frac{s}{4\epsilon}+O\left(\frac{1}{\epsilon^{\frac{3}{2}}}\right). \tag{8·27}$$

If instead of (6·14) we try the expansion

$$\lambda = -\frac{s}{\epsilon^{\frac{1}{2}}}+\tfrac{1}{2}q\frac{s}{2\epsilon}+O\left(\frac{1}{\epsilon^{\frac{3}{2}}}\right), \tag{8·28}$$

we find that no solution finite over the whole range of η exists.

To summarize, we have found three possible types of asymptotic forms for the eigenvalues and eigenfunctions when ϵ is large. Their properties may be identified as follows.

Type 1

The eigenvalue $\lambda = \sigma/2\Omega$ is given by

$$\lambda \sim \pm\frac{(2\nu+1)^{\frac{1}{2}}}{\epsilon^{\frac{1}{4}}}+\frac{s}{(4\nu+2)\,\epsilon^{\frac{1}{2}}} \quad (\nu=0,1,2,\ldots). \tag{8·29}$$

When $s>0$ the phase velocity $c=\sigma/s=2\Omega\lambda/s$ is towards the east or the west according as we take the positive or negative sign in the above equation. The northwards component of velocity v is equal to $-\mathrm{i}v^*/\sin\theta$ and so from (8·8)

$$v \sim -\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\eta^2}H_\nu(\eta)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \quad (\eta=\epsilon^{\frac{1}{4}}\mu). \tag{8·30}$$

Making use of equations (7·2) and the recurrence relations for the Hermite polynomials H_ν (see, for example, Morse & Feshbach 1953, p. 786) we have also

$$u \sim \pm\frac{1}{(2\nu+1)^{\frac{1}{2}}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}(\nu H_{\nu-1}+\tfrac{1}{2}H_{\nu+1})\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \tag{8·31}$$

and

$$\zeta^* \sim \mp\frac{1}{(2\nu+1)^{\frac{1}{2}}\epsilon^{\frac{1}{4}}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}(\nu H_{\nu-1}-\tfrac{1}{2}H_{\nu+1})\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{8·32}$$

It will be shown later that the kinetic energy of these motions exceeds the potential energy by a factor of 3.

Type 2

The eigenvalue $\lambda = \sigma/2\Omega$ is given by

$$\lambda \sim -\frac{s}{(2\nu'+1)\,\epsilon^{\frac{1}{2}}} \quad (\nu'=1,2,\ldots). \tag{8·33}$$

(It is assumed that $s>0$.) The zonal component of the phase velocity is given by

$$c = 2\Omega\lambda/s = -\frac{2\Omega}{(2\nu'+1)\,\epsilon^{\frac{1}{2}}} = -\frac{(gh)^{\frac{1}{2}}}{2\nu'+1}, \tag{8·34}$$

which is always towards the west. The northwards component of velocity is given by

$$v \sim -\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\eta^2}H_{\nu'}(\eta)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \quad (\eta=\epsilon^{\frac{1}{4}}\mu) \tag{8·35}$$

as in type 1, but from equations (7·2) and (8·33) we now find

$$u \sim \frac{2\nu'+1}{2s}\,\epsilon^{\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\left(H_{\nu'-1}-\frac{1}{2\nu'+2}H_{\nu'+1}\right)\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \tag{8·36}$$

and

$$\zeta^* \sim -\frac{2\nu'+1}{2s}\,\epsilon^{-\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\left(H_{\nu'-1}+\frac{1}{2\nu'+2}H_{\nu+1}\right)\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{8·37}$$

The kinetic energy in these motions is virtually all in the east–west component of velocity, and is equal asymptotically to the potential energy (see § 9).

Type 3

The eigenvalues $\lambda = \sigma/2\Omega$ are given by

$$\lambda \sim \frac{s}{\epsilon^{\frac{1}{2}}} + \frac{s}{4\epsilon}, \tag{8·38}$$

corresponding to a velocity

$$c = 2\Omega\lambda/s \sim 2\Omega\epsilon^{\frac{1}{2}} = (gh)^{\frac{1}{2}} \tag{8·39}$$

towards the east. The northwards component of velocity (after multiplying by -1) is given by

$$v \sim \mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\eta\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{8·40}$$

and the two other elements are given by

$$u \sim \frac{2}{s}\epsilon^{\frac{3}{4}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{8·41}$$

and

$$\zeta^* \sim \frac{2}{s}\epsilon^{\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{8·42}$$

From the value of the phase velocity, as well as from the proportionality of u and ζ^*, it can be seen that this solution represents a Kelvin wave travelling eastwards along the equator and presumably trapped by the Coriolis forces to north and south. The energy is divided almost equally between kinetic and potential.

From figures 1 to 6 we can now see that all of the above asymptotic forms are in fact realized. The forms of type 1 are found for every value of s (including $s = 0$), and for $\nu = 0, 1, 2, \ldots$. All those modes which for small values of ϵ are of the first class become for large values of ϵ waves of type 1. Further, among the westwards modes, the mode which for small ϵ is the lowest order mode of the second class also becomes for large ϵ a mode of type 1, namely the mode with $\nu = 0$.

When we come to consider the modes of type 2, we find that they exist for all positive integral values of ν'; the case $\nu' = 0$ is not found. All but one of the second-class modes become modes of type 2, with the exception of the one which becomes of type 1.

Lastly, the lowest of the eastwards-travelling modes, which for small ϵ is of the first class, becomes for large ϵ a Kelvin wave (type 3). It could be argued physically that the type 2 mode with $\nu' = 0$ cannot exist because if it did it would have the form of a Kelvin wave travelling in the wrong direction, i.e. towards the west; whereas if a Kelvin wave is to be trapped at the equator by Coriolis forces it clearly must travel towards the east.

All three types of limiting form can be derived by using the equatorial β-plane approximation introduced by Rattray (1964). In particular solutions of types 1 and 2, in which $v \propto \mathrm{e}^{-\frac{1}{2}\eta^2}H_{\nu'}(\eta)$ can be shown to exist by this method. However, it is not so obvious why the type 2 solutions with $\nu' = 0$ cannot occur*. Rattray also noted the existence, in his approximation, of a class of Kelvin waves in which $v = 0$ identically. In our approximation v is small but not zero.

* *Note added in proof*, 10 *October* 1967. The author's attention has been drawn to a recent paper by Matsuno (1966) in which the equatorial β-plane approximation is successfully applied. In particular, the case $\nu' = 0$ is fully discussed.

What is the significance of the integers ν and ν'? We have seen that ν is equal to the degree of the polynomial $H_\nu(\eta)$. Hence ν or ν' represents the number of nodal lines of v in the open interval $-1 < \mu < 1$ for large values of ϵ.

Generally we may define the 'signature' Σ of any particular mode, at a given value of ϵ, as the number of nodal lines of v in the open interval $-1 < \mu < 1$. Thus for large values of ϵ we have $\Sigma = \nu$ or ν'.

Consider the behaviour of Σ as $\epsilon \to 0$. From the relation

$$v = -\frac{\partial \Phi}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \phi},$$

and the two asymptotic forms (4·6) and (4·10) it follows that for waves of the *first class* v is asymptotically proportional to $\mathrm{d}P_n^s/\mathrm{d}\mu$ and for waves of the *second class* it is proportional to $P_n^s(\mu)$. But P_n^s always has $(n-s)$ zeros in the open interval $-1 < \mu < 1$, and generally $\mathrm{d}P_n^s/\mathrm{d}\mu$ has $(n-s+1)$ zeros (by Rolle's theorem); except that when $s = 0$, $\mathrm{d}P_n^s/\mathrm{d}\mu$ has only $(n-1)$ zeros. Hence for waves of the first class

$$\Sigma \sim \begin{cases} (n-s+1) & (s \geqslant 1), \\ (n-1) & (s = 0), \end{cases}$$

as $\epsilon \to 0$, and for waves of the second class

$$\Sigma \sim (n-s).$$

From figures 1 to 6 it will be seen that in the case of the *westwards* travelling waves, whether of the first or the second class, the signature is the same at the end of each curve. On the other hand, for the *eastwards* travelling modes (when $s > 0$) the signature generally changes; the curves corresponding to $\nu = 0, 1, 2, \ldots$ at small values of $\epsilon^{-\frac{1}{2}}$ have $(n-s+1) = 1, 2, 3, \ldots$ respectively at large values of $\epsilon^{-\frac{1}{2}}$. Thus there is at least one change of signature in the range of the parameter ϵ.

In the exceptional case, the lowest symmetric mode, which has signature $(n-s+1) = 1$ as $\epsilon^{-\frac{1}{2}} \to \infty$, becomes the Kelvin wave, also with signature 1, as $\epsilon^{-\frac{1}{2}} \to 0$.

9. The eigenfunctions for positive ϵ

Before the eigenfunctions are presented one must determine a suitable method of normalization. If one were considering only the surface elevation ζ or the corresponding pressure fluctuation it might be appropriate to normalize so as to make the integral of ζ^2 over $-1 < \mu < 1$ equal to a constant, say unity. This would be equivalent to assuming the total *potential* energy over the sphere to be a constant. However, we wish to include also the two components of velocity u and v. In some limiting forms the kinetic energy may greatly exceed the potential energy. We shall therefore normalize so as to make the total energy, kinetic plus potential, equal to a constant. Thus we assume

$$\iint [\tfrac{1}{2}\rho h(u^2+v^2) + \tfrac{1}{2}\rho g \zeta^2]\,\mathrm{d}S = 4\pi E, \tag{9·1}$$

where E denotes the mean energy density per unit area of the sphere.

Now the two quantities q_0 and ζ_0 defined by

$$q_0 = (8E/\rho h)^{\frac{1}{2}}, \qquad \zeta_0 = (8E/\rho g)^{\frac{1}{2}} \tag{9·2}$$

are typical units of velocity and of vertical displacement. Hence we may form the non-dimensional variables

$$u' = u/q_0, \quad v' = v/q_0, \quad \zeta' = \zeta/\zeta_0. \tag{9·3}$$

In terms of these the total energy (9·1) becomes

$$4\pi E = 4E\iint (u'^2+v'^2+\zeta'^2)\,\mathrm{d}S. \tag{9·4}$$

In other words $(u'^2+v'^2)$ and ζ'^2 represent the relative contributions to the densities of kinetic and potential energy, and moreover

$$\iint (u'^2+v'^2+\zeta'^2)\,\mathrm{d}S = \pi. \tag{9·5}$$

In the following we shall display the dependence of u', v', ζ' on the colatitude θ by letting

$$u' = U\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \quad v' = V\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \quad \zeta' = Z\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{9·6}$$

and plotting the three functions U, V, Z against θ between 0 and $\frac{1}{2}\pi$. The total energy is independent of the time t, so that on taking mean values with respect to t in equation (9·5) we have

$$\tfrac{1}{2}\iint (U^2+V^2+Z^2)\,\mathrm{d}S = \pi. \tag{9·7}$$

The integrand is also independent of ϕ, and since

$$\mathrm{d}S \sim \mathrm{d}\mu\,\mathrm{d}\phi \sim \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{9·8}$$

we have from equation (9·7)

$$\int_{-1}^{1} (U^2+V^2+Z^2)\,\mathrm{d}\mu = 1, \tag{9·9}$$

or equivalently

$$\int_{0}^{\pi} (U^2+V^2+Z^2)\sin\theta\,\mathrm{d}\theta = 1. \tag{9·10}$$

Now u, v and ζ are given in terms of Φ and Ψ by equations (3·1) and (3·13). Hence an expression for the total energy equivalent to (9·1) is

$$\iint [\tfrac{1}{2}\rho g(h^2/\sigma^2)(\nabla^2\Phi)^2 - \tfrac{1}{2}\rho h(\Phi\nabla^2\Phi+\Psi\nabla^2\Psi)]\,\mathrm{d}S = 4\pi E. \tag{9·11}$$

(Green's theorem has been used in the derivation of the second term.) On expressing Φ and Ψ in terms of the series (3·15) we obtain

$$\tfrac{1}{2}\pi\rho h \sum_{n=s}^{\infty}\left[\frac{n^2(n+1)^2}{\epsilon\lambda^2}A_n^{s2}+n(n+1)\,(A_n^{s2}+B_n^{s2})\right]\frac{1}{n+\frac{1}{2}}\frac{(n+s)!}{(n-s)!} = 4\pi E, \tag{9·12}$$

or from (9·2)

$$\sum_{n=s}^{\infty}\left[\frac{n^2(n+1)^2}{\epsilon\lambda^2}A_n^{s2}+n(n+1)\,(A_n^{s2}+B_n^{s2})\right]\frac{1}{n+\frac{1}{2}}\frac{(n+s)!}{(n-s)!} = q_0^2. \tag{9·13}$$

For the purpose of calculation q_0, the unit of velocity, was taken equal to unity and the eigenvectors were normalized by means of equation (9·13), with $q_0 = 1$.

The three quantities u, v, ζ are given in terms of the calculated functions U, V, Z by the relations

$$\left.\begin{aligned} u &= q_0\,U\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \\ v &= q_0\,W\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \\ \zeta &= q_0(h/g)^{\frac{1}{2}}Z\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \end{aligned}\right\} \tag{9·14}$$

where q_0 is an arbitrary velocity. The mean energy density E over the surface of the sphere is given by

$$E = \tfrac{1}{8}\rho h q_0^2. \tag{9·15}$$

Some calculated eigenfunctions U, V and Z are shown in figure 7 for $s = 0$, and in figures 8 to 10 for $s = 1$ and figures 11 to 13 for $s = 2$. Each mode has been calculated at four different values of ϵ, namely† $\epsilon = 1$, 10, 10^2 and 10^3, so that the behaviour of each mode as ϵ varies can be followed. As ϵ increases, the concentration of the energy towards the equator ($\theta = 90°$) is apparent. It can be seen also that in the eastwards modes (figures 8 and 11) the number of zeros of V remains invariant, whereas in the westwards modes of the first class (figures 9 and 12) an extra zero appears as ϵ increases. In the westwards modes of the second class (figures 10 and 13) the number of zeros of V also remains invariant.

The development of the Kelvin wave as ϵ increases can be seen in the lowest mode $(n-s) = 0$ in figures 8 and 11. In each case the northwards component of velocity V is becoming small compared with the eastwards component U.

The eigenfunctions were calculated at intervals of 1° from $\theta = 0°$ to $\theta = 90°$. The numerical results were checked in three ways.

(1) The substitution

$$\left.\begin{aligned}\zeta^* &= (1-\mu^2)^{s/2\lambda}\,X,\\ v^* &= (1-\mu^2)^{-s/2\lambda}\,Y,\end{aligned}\right\} \tag{9·16}$$

reduces equation (7·4) to the canonical form

$$\left.\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}\mu} &= \frac{\lambda^2-\mu^2}{\lambda(1-\mu^2)^{1+s/\lambda}}\,Y,\\ \frac{\mathrm{d}Y}{\mathrm{d}\mu} &= \frac{s^2-\epsilon\lambda^2(1-\mu^2)}{\lambda(1-\mu^2)^{1-s/\lambda}}\,X.\end{aligned}\right\} \tag{9·17}$$

By computing X and Y it was verified that the zeros of X were in fact among the turning-points of Y, and vice versa.

(2) The integral (9·10) was calculated by Simpson's rule, verifying the correct value to one part in 10^5.

(3) Convergence of the individual values of the function to at least four figures, and usually to 8, was verified by comparing the results for $N = 20$ with those for $N = 30$, or those for $N = 40$ with those for $N = 50$.

For each mode the ratio of the kinetic energy to the total energy was also calculated, and the results are shown in figure 14 (for $s = 0$) and figure 15 (for $s = 1, 2$). At small values of ϵ (large values of $\sqrt{(gh)}/2\Omega$) the ratio k.e./(k.e.+p.e.) tends to 0·5 for waves of the first class and 1·0 for waves of the second class. However, at large values of ϵ (small values of $\sqrt{(gh)}/2\Omega$) the ratio tends to 0·75 for waves of type 1 and 0·5 for waves of types 2 and 3.

These results can be verified analytically as follows. Since from § 8 the energy is evidently concentrated near the equator the integral in (9·1) becomes

$$\iint[\tfrac{1}{2}\rho h(u^2+v^2)+\tfrac{1}{2}\rho g\zeta^2]\,\epsilon^{-\frac{1}{4}}\,\mathrm{d}\eta\,\mathrm{d}\phi. \tag{9·18}$$

† The corresponding values of λ were found by interpolation; see table 5. These have been checked by solving numerically for the corresponding values of η. They are correct to the number of places given.

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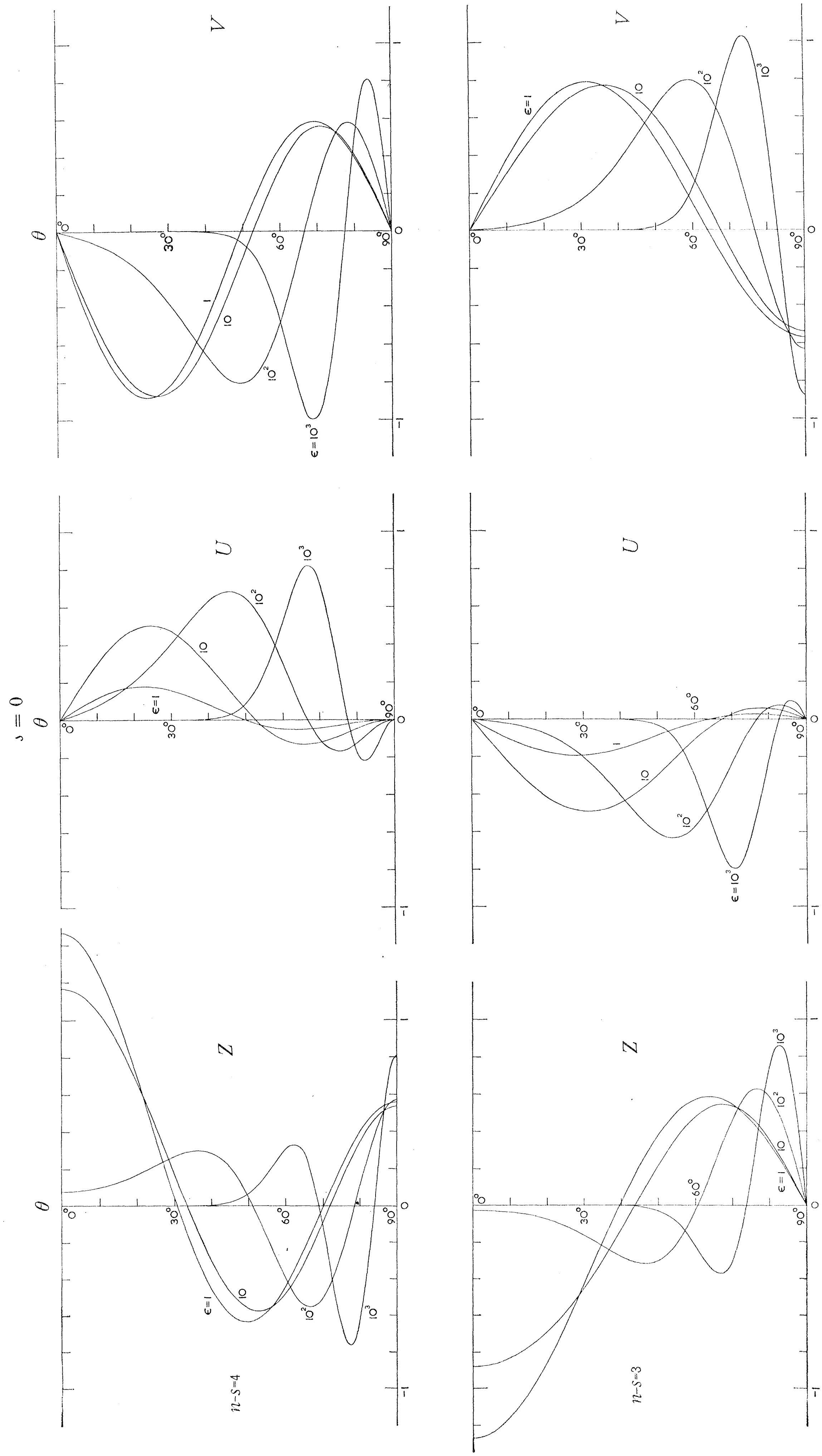

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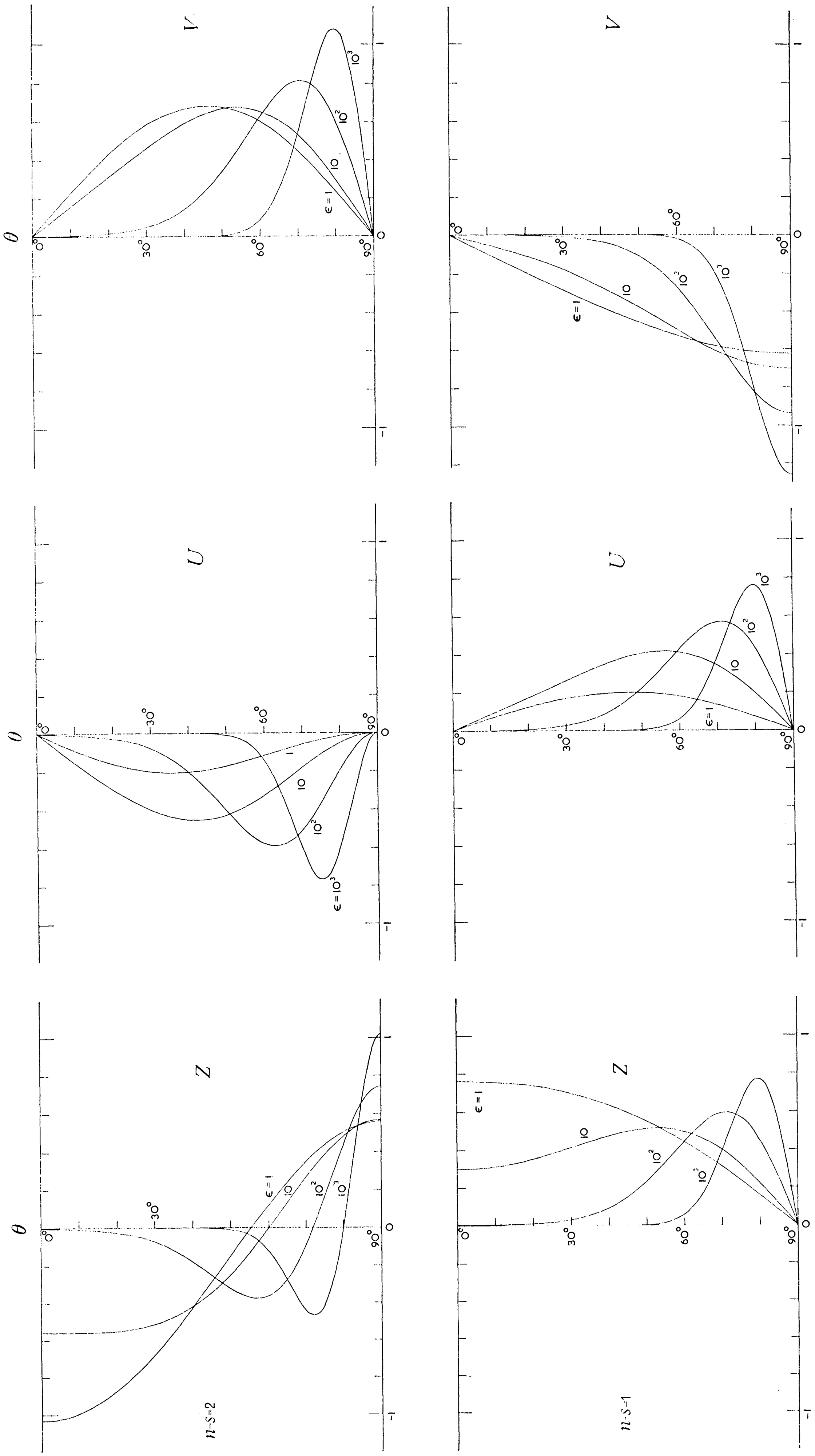
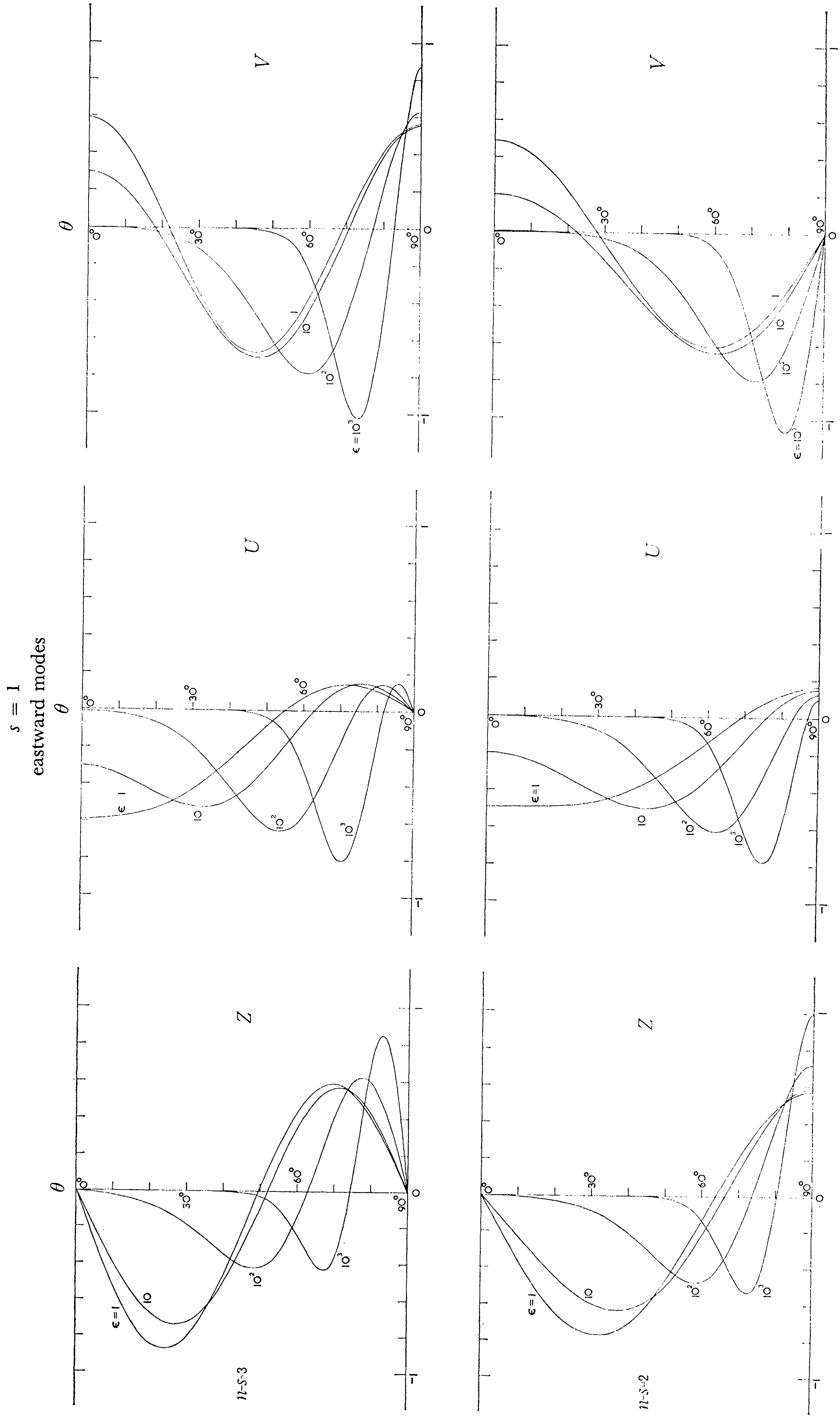


FIGURE 7. Eigenfunctions Z, U and V when $s = 0$: the four lowest modes.


s = 1
eastward modes
V
U
Z
θ

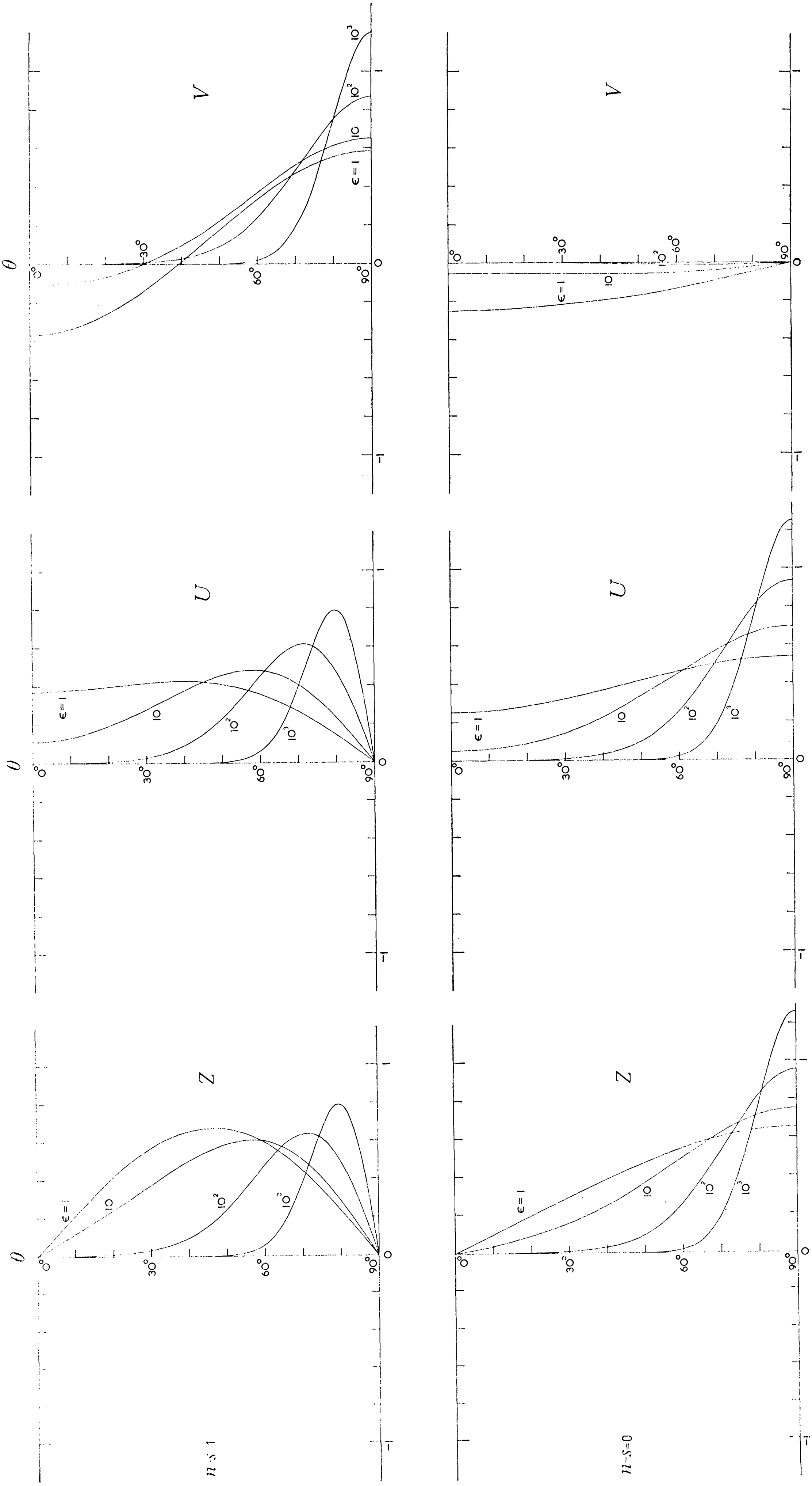
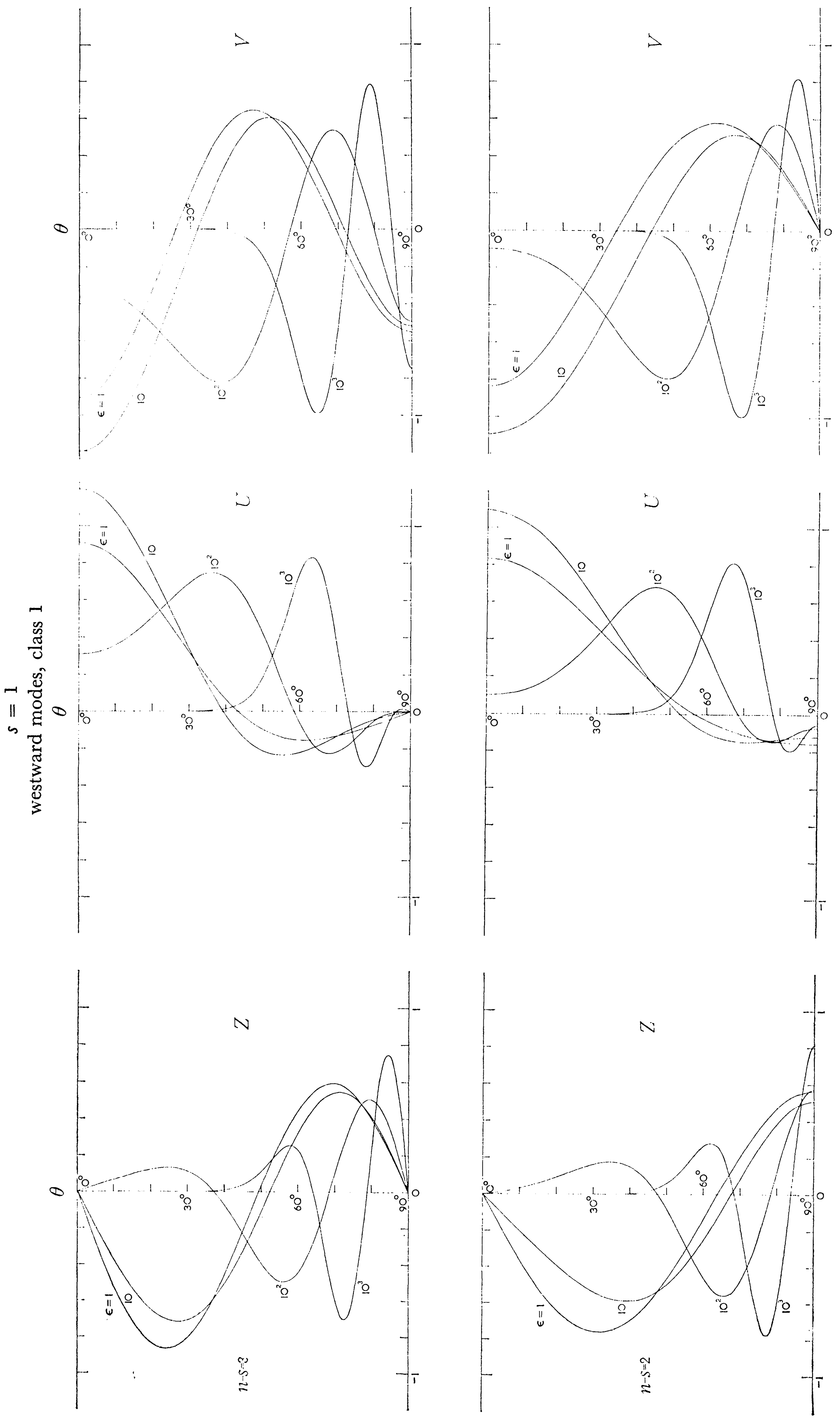


FIGURE 8. Eigenfunctions Z, U and V when $s = 1$: the four lowest eastwards modes.



s = 1
westward modes, class 1

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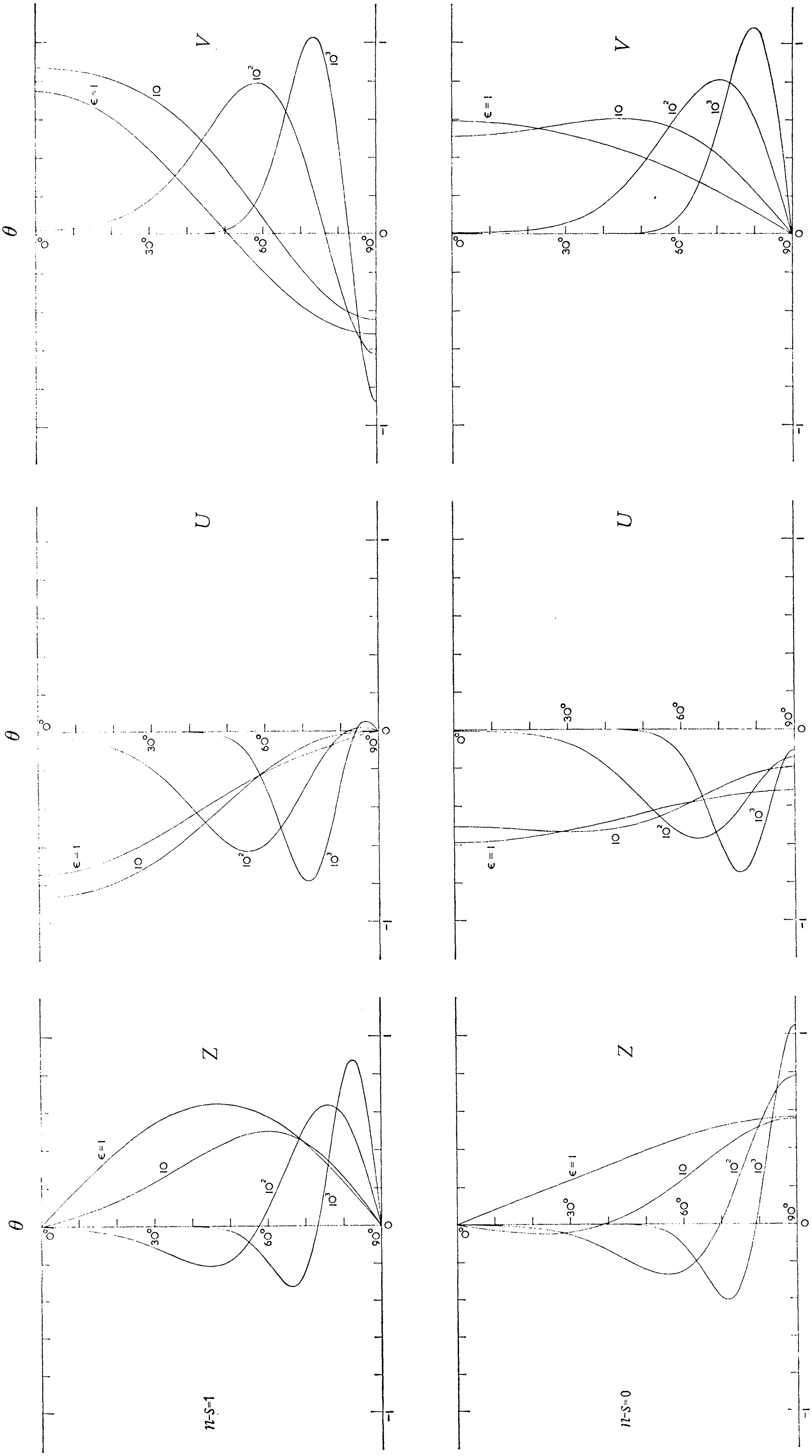

FIGURE 9. Eigenfunctions Z, U and V when $s = 1$: the four lowest westwards modes of class 1.

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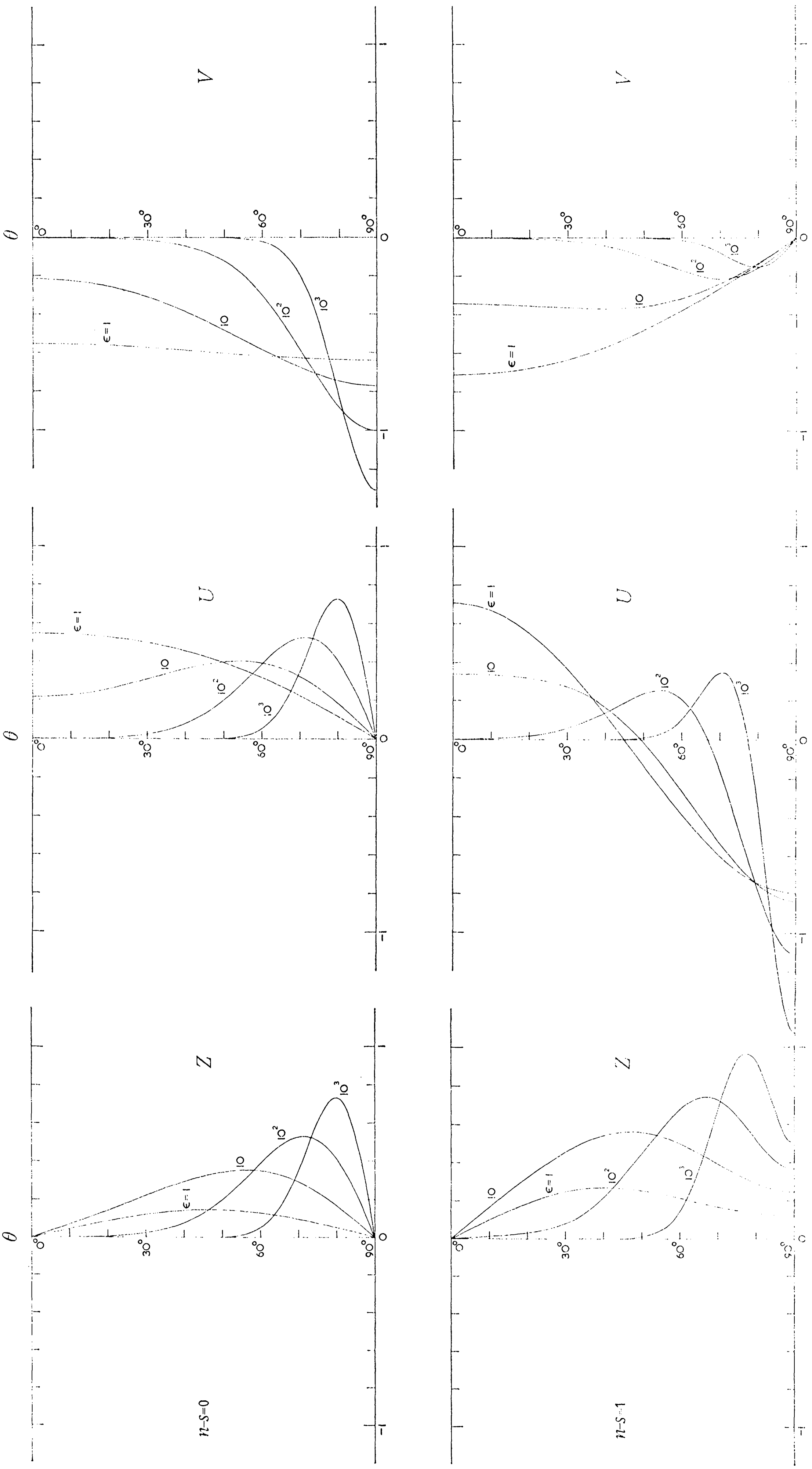

$s = 1$
westward modes, class 2

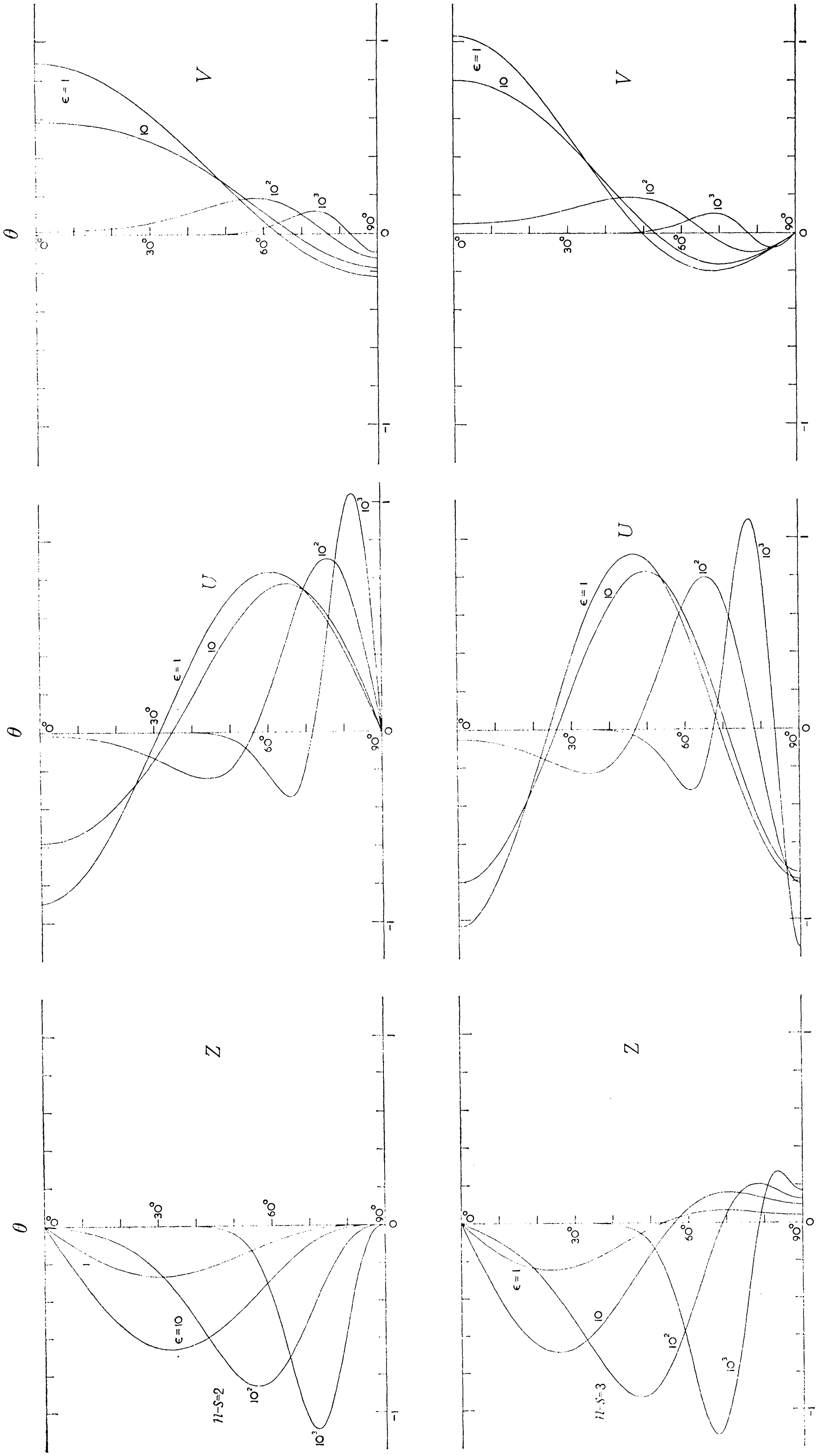

FIGURE 10. Eigenfunctions Z, U and V when $s = 1$: the four lowest westwards modes of class 2.

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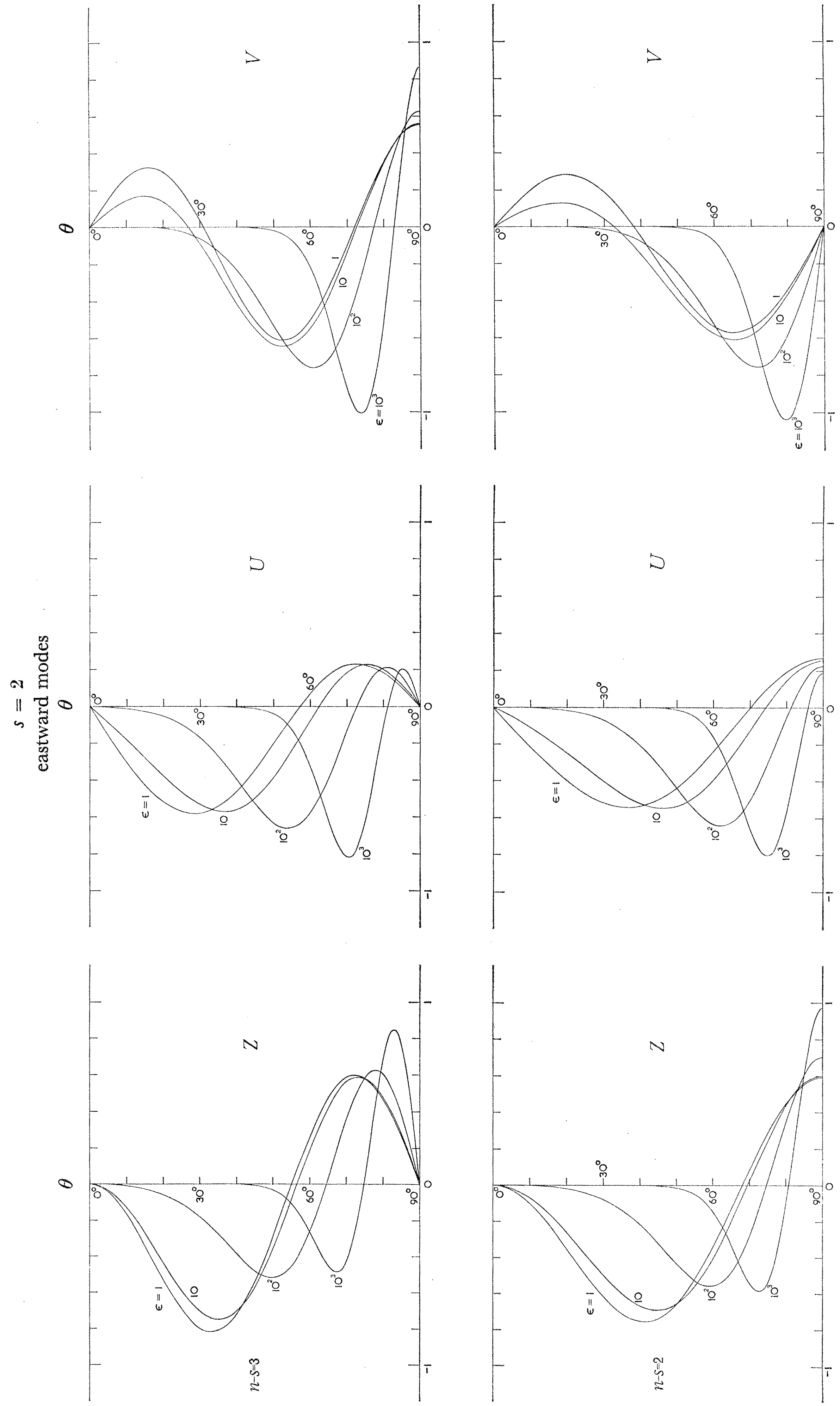
s = 2
eastward modes
V
U
Z
θ
$n-s=3$
$n-s=2$
ε = 1
10
10²
ε = 10³
0°
30°
60°
90°

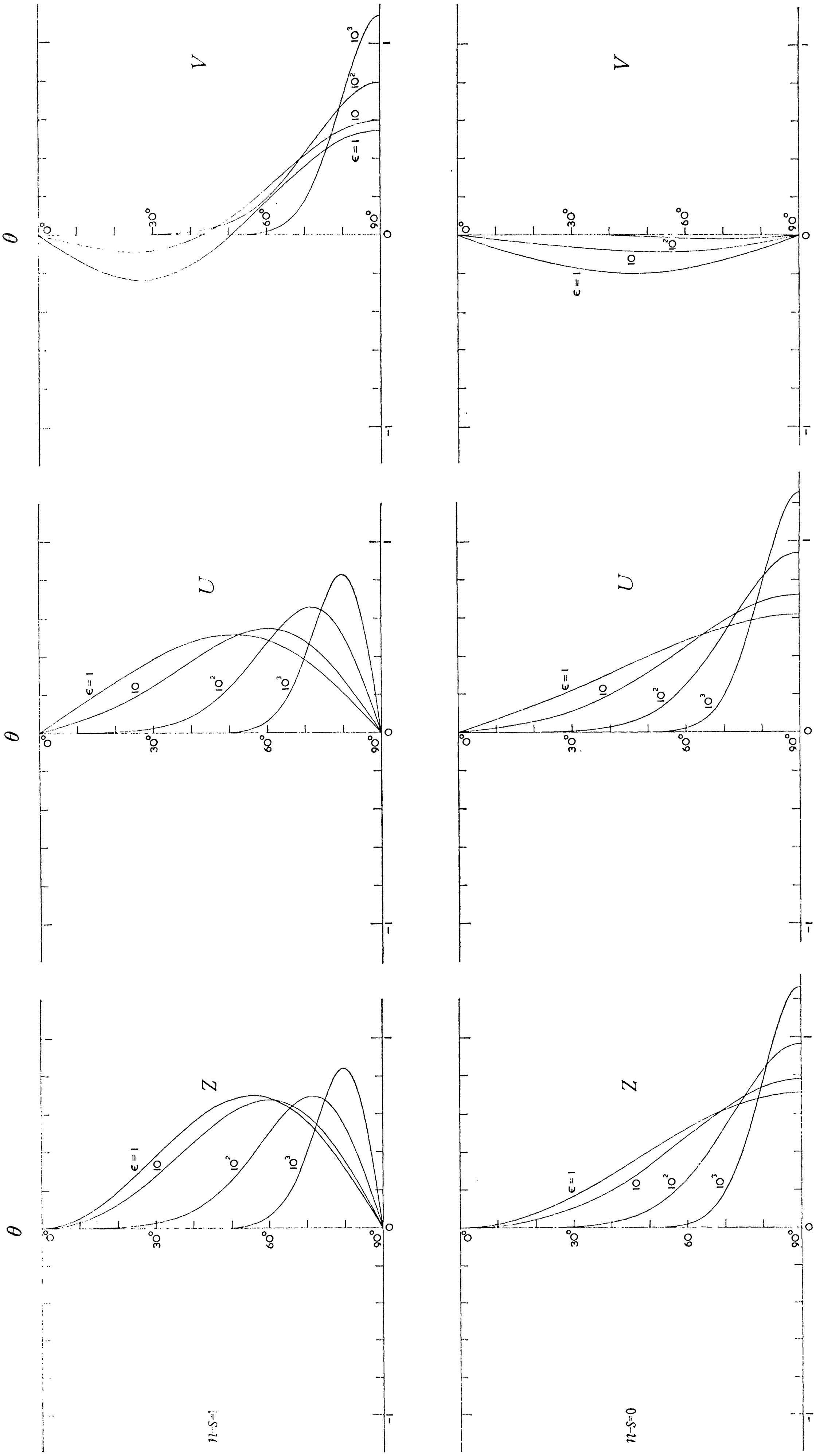

FIGURE 11. Eigenfunctions Z, U and V when $s = 2$: the four lowest eastwards modes.

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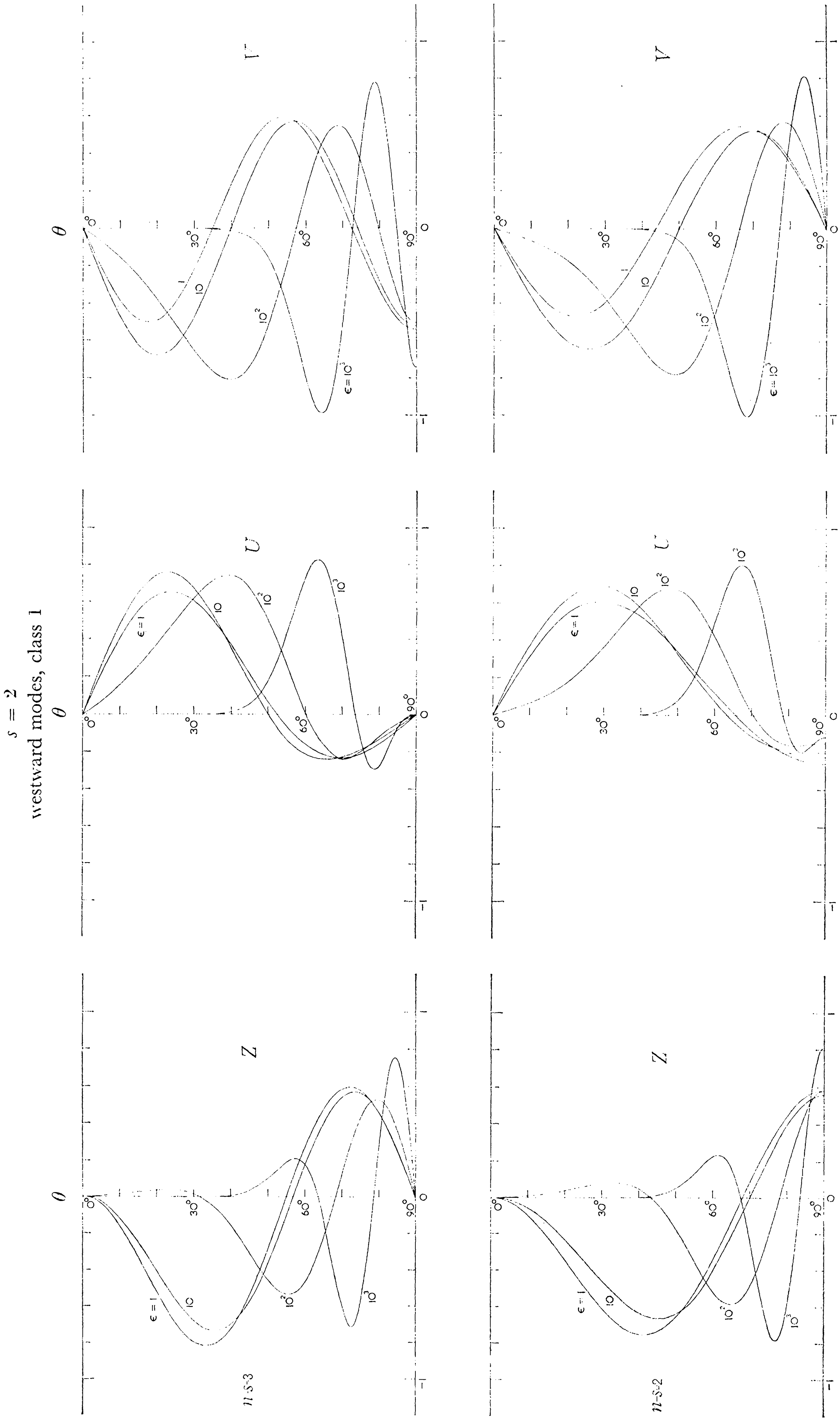

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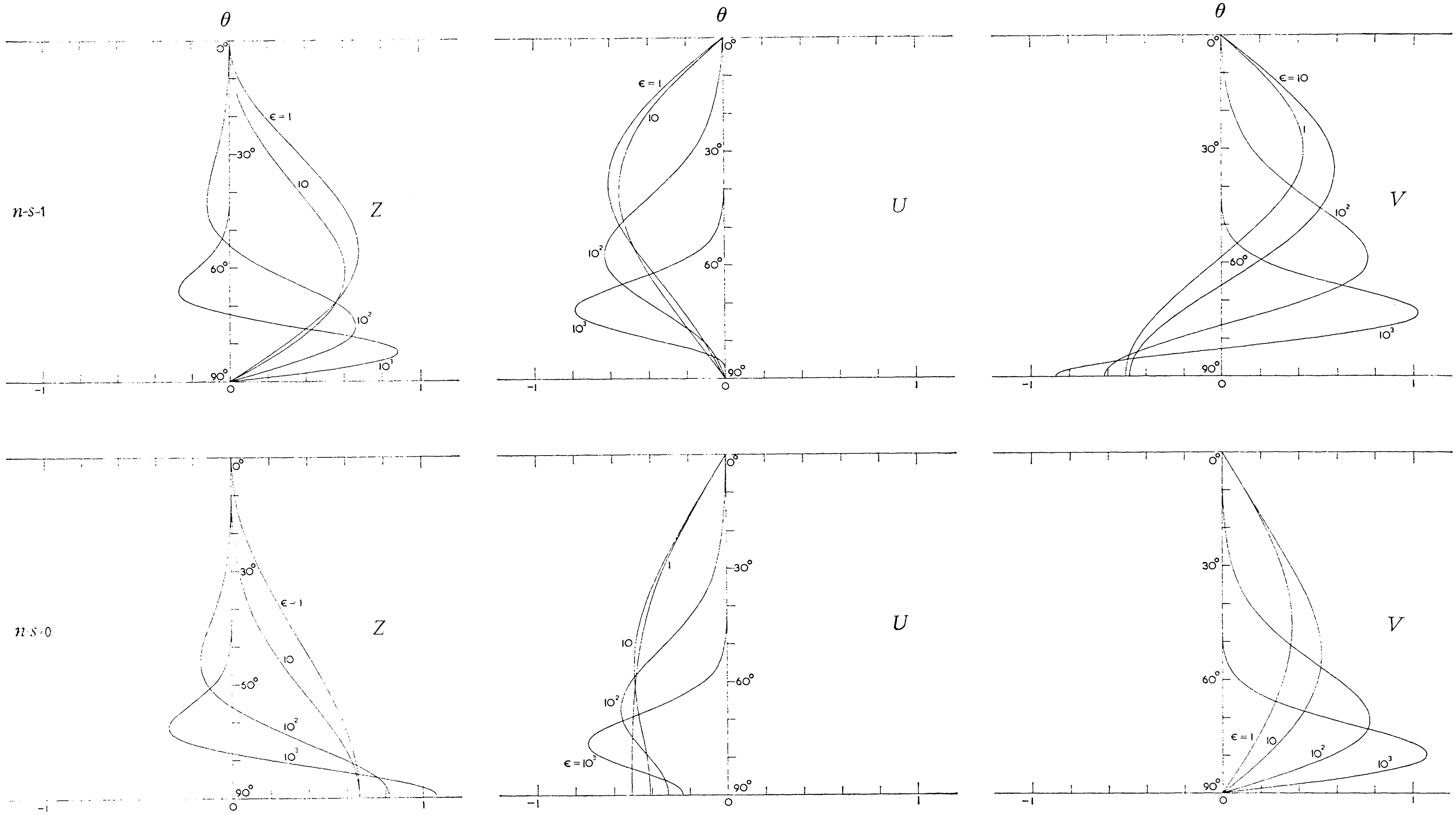
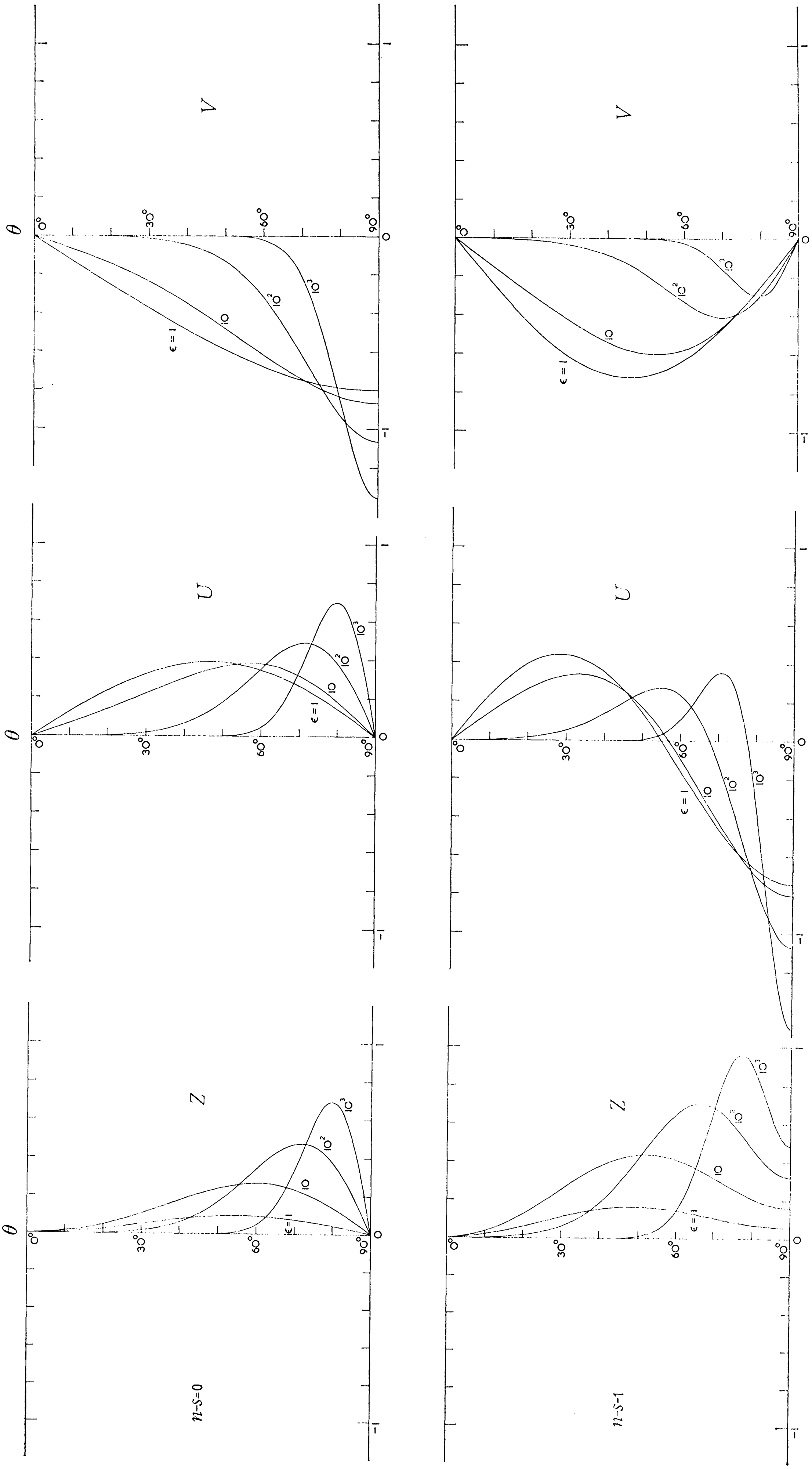


FIGURE 12. Eigenfunctions Z, U and V when $s = 2$: the four lowest westward modes of class 1.



$s = 2$
westward modes, class 2

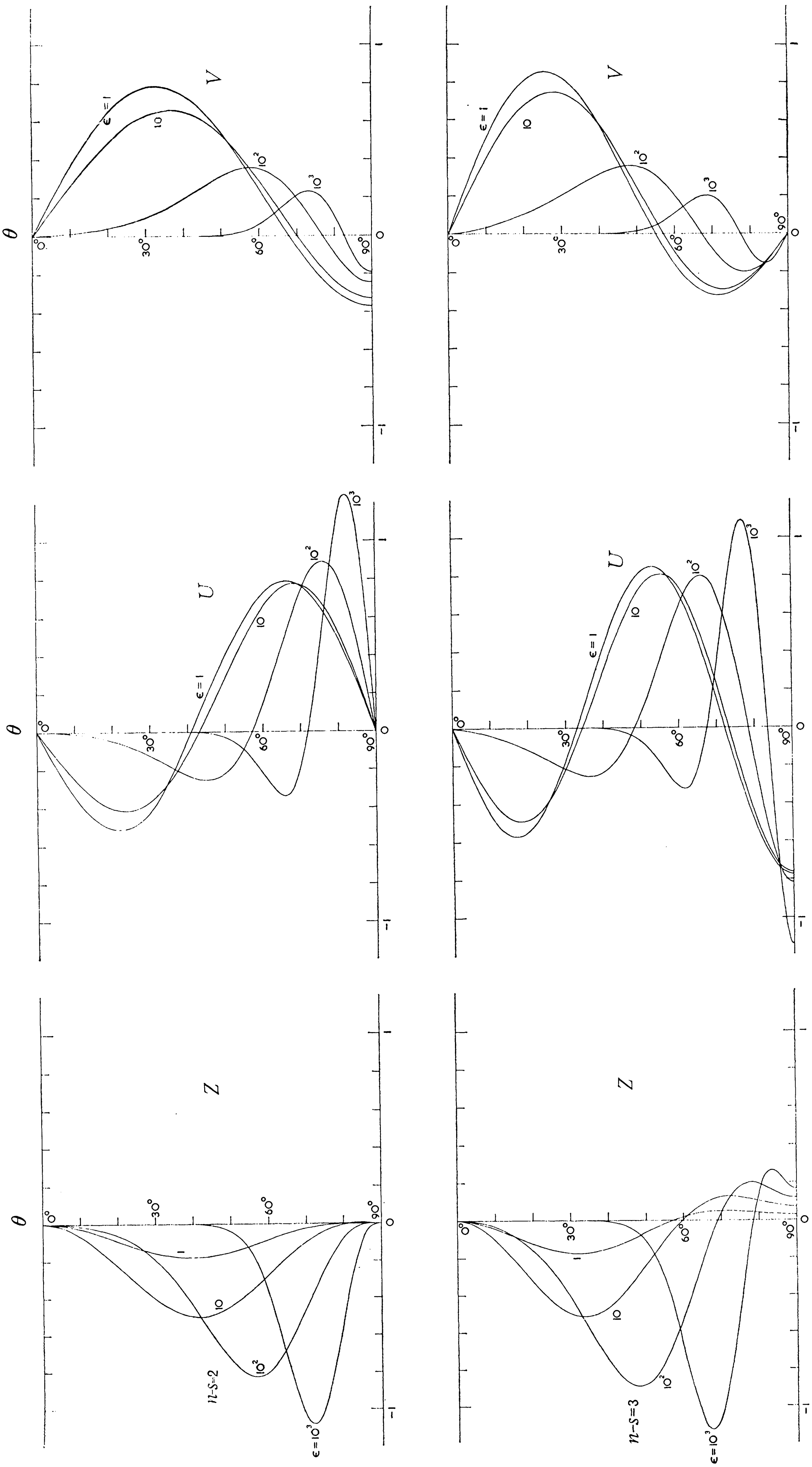


FIGURE 13. Eigenfunctions Z, U and V when $s = 2$: the four lowest westwards modes of class 2.

On replacing each term by its mean value with respect to time and noting that $g\zeta^2 = \epsilon h\zeta^{*2}$ we obtain

$$\tfrac{1}{2}\pi\rho h\epsilon^{-\frac{1}{4}}\int_{-\infty}^{\infty}[|u^2|+|v^2|+\epsilon|\zeta^{*2}|]\,\mathrm{d}\eta. \tag{9·19}$$

On substituting from equations (8·30) to (8·32) and evaluating each term separately, using the identity

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\eta^2}H_\nu(\eta)\,H_{\nu'}(\eta)\,\mathrm{d}\eta = \begin{cases}2^\nu\nu!\,\pi & (\nu'=\nu),\\ 0 & (\nu'\neq\nu),\end{cases} \tag{9·20}$$

we find

$$\left.\begin{aligned}\int_{-\infty}^{\infty}|u^2|\,\mathrm{d}\eta &= 2^{\nu-1}\nu!\,\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}|v^2|\,\mathrm{d}\eta &= 2^{\nu}\nu!\,\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}\epsilon\,|\zeta^{*2}|\,\mathrm{d}\eta &= 2^{\nu-1}\nu!\,\pi^{\frac{1}{2}}.\end{aligned}\right\} \tag{9·21}$$

Thus we see that in waves of type 1 the kinetic energy in the eastwards component, the kinetic energy in the northwards component and the potential energy are in the ratios 1:2:1. The ratio of the total kinetic energy to the potential energy is therefore 3:1.

In waves of type 2 we have from equations (8·35) to (8·37)

$$\left.\begin{aligned}\int_{-\infty}^{\infty}|u^2|\,\mathrm{d}\eta &= \frac{(2\nu'+1)^3\,2^{\nu'-3}(\nu'-1)!}{(\nu'+1)\,s^2}\,\epsilon^{\frac{1}{2}}\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}|v^2|\,d\eta &= 2^{\nu'}\nu'!\,\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}\epsilon\,|\zeta^{*2}|\,\mathrm{d}\eta &= \frac{(2\nu'+1)^3\,2^{\nu'-3}(\nu'-1)!}{(\nu'+1)\,s^2}\,\epsilon^{\frac{1}{2}}\pi^{\frac{1}{2}},\end{aligned}\right\} \tag{9·22}$$

so that the kinetic energy in the northwards component is relatively small. The total kinetic energy is almost equal to the potential energy, and each is about half the total energy.

In waves of type 3, since $\eta = \frac{1}{2}H_1$, we have from equations (8·40) to (8·42)

$$\left.\begin{aligned}\int_{-\infty}^{\infty}|u^2|\,\mathrm{d}\eta &= \frac{4}{s^2}\epsilon^{\frac{3}{2}}\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}|v^2|\,\mathrm{d}\eta &= \tfrac{1}{2}\pi^{\frac{1}{2}},\\ \int_{-\infty}^{\infty}\epsilon\,|\zeta^{*2}|\,\mathrm{d}\eta &= \frac{4}{s^2}\epsilon^{\frac{3}{2}}\pi^{\frac{1}{2}}.\end{aligned}\right\} \tag{9·23}$$

Hence the energy in the northwards component is very much smaller than that in the eastwards component, and the kinetic energy is half the total energy as in type 2.

[*Note added* 6 *February* 1967]

I am indebted to one of the referees for the following interpretation of these results. It is well known that in a non-rotating system the kinetic energy of a free oscillation about absolute equilibrium must be equal to the potential energy. The generalization to a rotating system (see, for example, Lamb 1932, p. 315) is that

$$\text{k.e.}+\Omega M' = \text{p.e.}, \tag{9·24}$$

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where k.e. and p.e. are the kinetic and potential energies as defined in § 2 (by the definition of g, the quantity p.e. includes also the potential energy of rotation) and where M' is given by

$$M' = \Sigma m(\xi\dot{\eta} - \dot{\xi}\eta). \tag{9·25}$$

Here m denotes the mass of each particle; ξ and η are the displacements referred to rectangular axes perpendicular to the rotation axis, and the summation Σ is over all the particles. The term $\Omega M'$ in (9·24) can be interpreted as the perturbation of the kinetic energy of rotation. Let us evaluate this quantity.

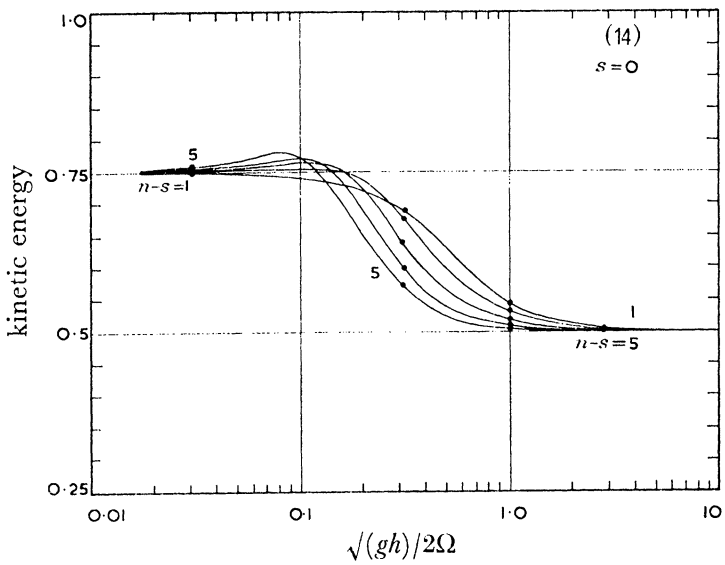


FIGURE 14. The ratio of kinetic energy to total energy when $s = 0$.

Since the expression $(\xi\dot{\eta} - \dot{\xi}\eta)$ is invariant with respect to orientation of the axes in the equatorial plane we may take ξ locally eastwards, and then η is directed towards the axis of rotation. To first order we have

$$\dot{\xi} = u, \quad \dot{\eta} = v\cos\theta \tag{9·26}$$

so that to second order M' may be written

$$M' = \iint \rho h[v\int u\,\mathrm{d}t - u\int v\,\mathrm{d}t]\cos\theta\,\mathrm{d}S. \tag{9·27}$$

From this expression it can be seen that M' will tend to be negative whenever eastward motions go with polewards displacements and westwards motions go with equatorial displacements. This will then allow the kinetic energy to exceed the potential energy.

Taking mean values with respect to time in (9·27) and integrating with respect to the longitude ϕ we have

$$M' = 2\pi\rho h\,\mathscr{R}\int_{-1}^{1}\frac{u^*v}{\mathrm{i}\sigma}\mu\,\mathrm{d}\mu, \tag{9·28}$$

where $\mathscr{R}$ denotes the real part and an asterisk denotes the complex conjugate. On substitution in this formula from equations (8·29) to (8·32) we find, for the waves of type 1,

$$\Omega M' = -\tfrac{1}{2}\pi^{\frac{3}{2}}\frac{\rho h}{\epsilon^{\frac{1}{4}}}2^{\nu}\nu!. \tag{9·29}$$

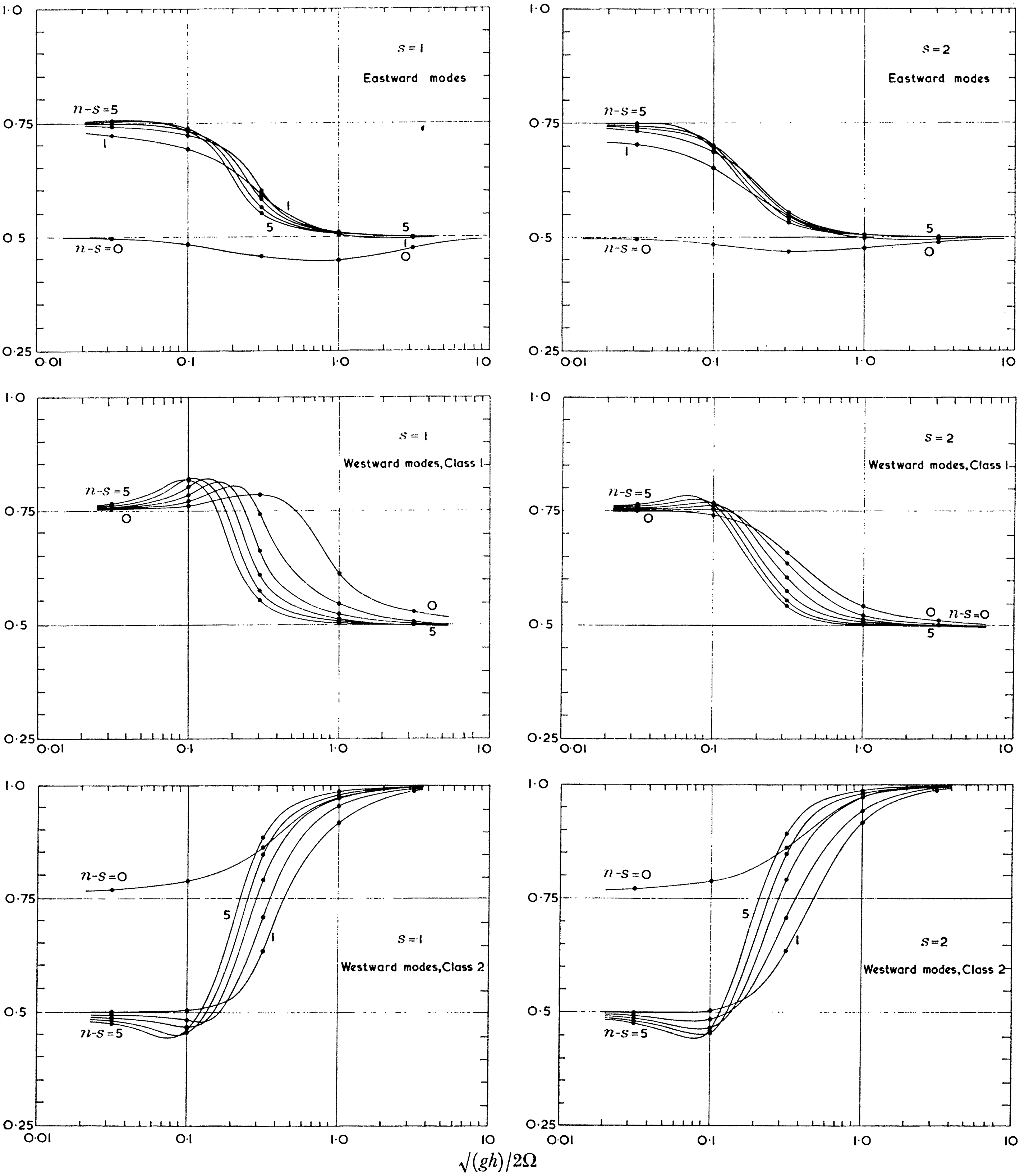


FIGURE 15. The ratio of kinetic energy to total energy when $s = 1$ and 2.

This is equal to -2 times the kinetic energy in the eastwards component of motion. The relation (9·24) is thus verified.

However, in the waves of types 2 and 3, the north–south displacements are relatively small. Hence $\Omega M'$ is negligible compared to the total kinetic energy, and so the kinetic and potential energies must be equal.

10. The eigenvalues for negative ϵ

The method of computation described in § 5 revealed also the existence of eigenvalues corresponding to negative values of ϵ. These eigenvalues and the corresponding modes are undoubtedly necessary for the complete expansion of any arbitrary function in terms of the eigenfunctions of the Laplace equations. Moreover, as we shall see in § 13, they can correspond physically to *forced* motions in a stably stratified fluid.

Leaving aside until § 13 the application to forced oscillations, we shall find it convenient to consider for the present the eigenfunctions as representing free oscillations in an *unstably* stratified fluid, or in a fluid with negative depth h. Thus we shall think of these oscillations as taking place in a thin fluid layer of thickness $-h$ on the inside of a rotating spherical shell in the presence of a gravitational acceleration towards the centre.

It may be noted that for free oscillations the energy equation (2·5) remains valid even when h is negative. Thus I_1+I_2 remains a constant independent of time. On the other hand, I_1 and I_2, as defined by (2·6) should now be interpreted as minus the kinetic energy and minus the potential energy respectively. These two are of opposite sign. Thus the kinetic energy is positive and the potential energy is negative.

The eigenvalues of the lowest modes are given in tables 3 and 4 and shown graphically in figures 16 to 21. Typical curves can be seen in figure 17, which corresponds to $s = 1$. All the eigenvalues are less than unity in absolute value, so that the periods are all greater than 12 h. Those eigenvalues corresponding to westward travelling modes (figure 17 *b*) tend to finite values as $\epsilon \rightarrow -0$, which are in fact the same limiting values as in figure 2 *b*, when $\epsilon \rightarrow +0$. However as $-\epsilon$ increases the eigenvalues come together in pairs and when $\epsilon \rightarrow -\infty$ the eigenvalues all tend to -1.

Figure 17 *a* shows that the eastwards travelling modes likewise have a limiting value of -1, but that there is a minimum value of $-\epsilon$ for which any given mode can exist. The eigenvalues run together very closely in pairs, and there is apparently another asymptote given by $\epsilon\lambda = -s$.

These new limiting cases will now be investigated.

11. Asymptotic forms of the solutions as $\epsilon \rightarrow -\infty$

The equation for v^*, namely equation (6·8), after division by λ becomes

$$\left[(\nabla^2 - s/\lambda) - \frac{2\epsilon\lambda^2\mu}{s^2-\epsilon\lambda^2(1-\mu^2)}(D-s\mu/\lambda)+\epsilon(\lambda^2-\mu^2)\right]v^* = 0. \tag{11·1}$$

We consider the case when ϵ is large and negative. Suppose first that $\lambda \doteqdot -1$. From (11·1) it appears that $(\lambda^2-\mu^2)$ must be small, hence $(1-\mu^2)$ is small and so the energy is concentrated near the poles. Consider the neighbourhood of the north pole, by writing

$$\omega = (-\epsilon)^{\frac{1}{4}}\theta, \tag{11·2}$$

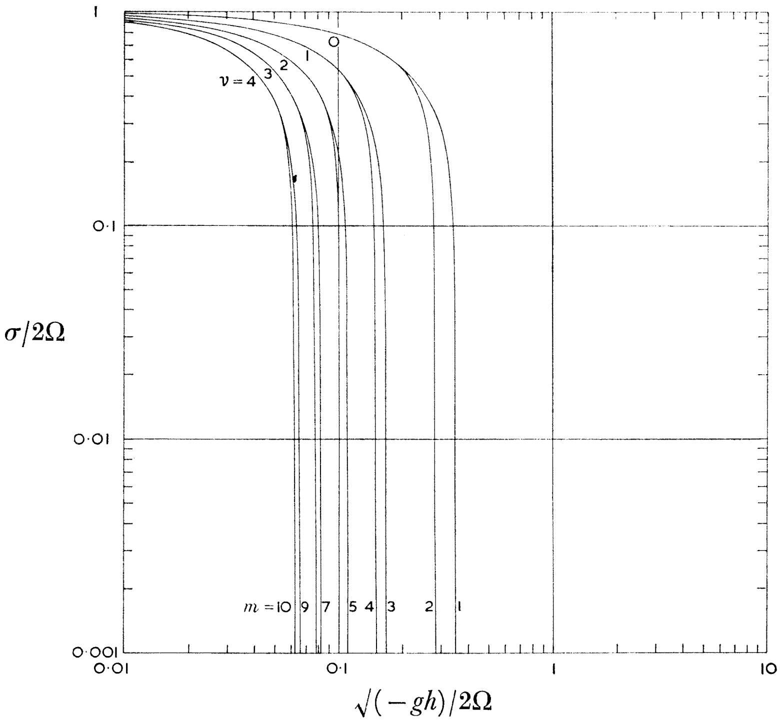


FIGURE 16. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 0$.

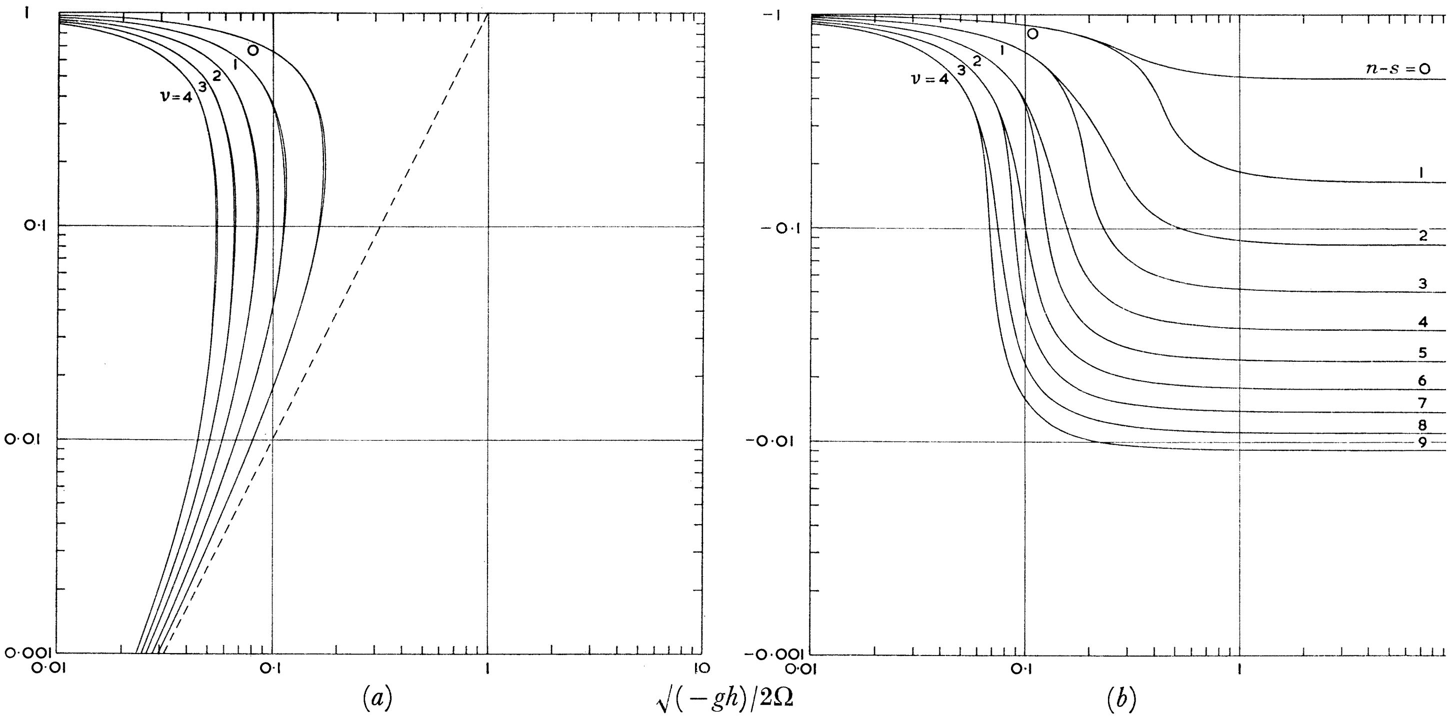
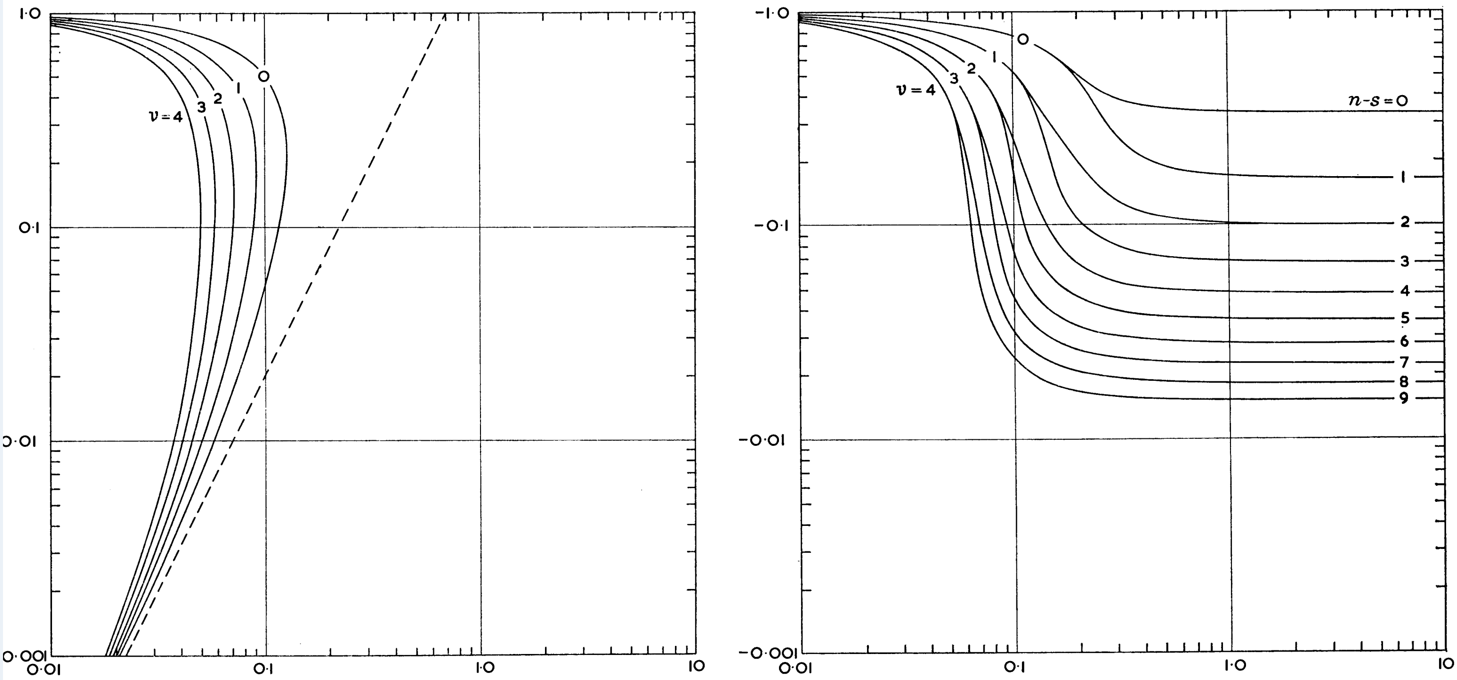
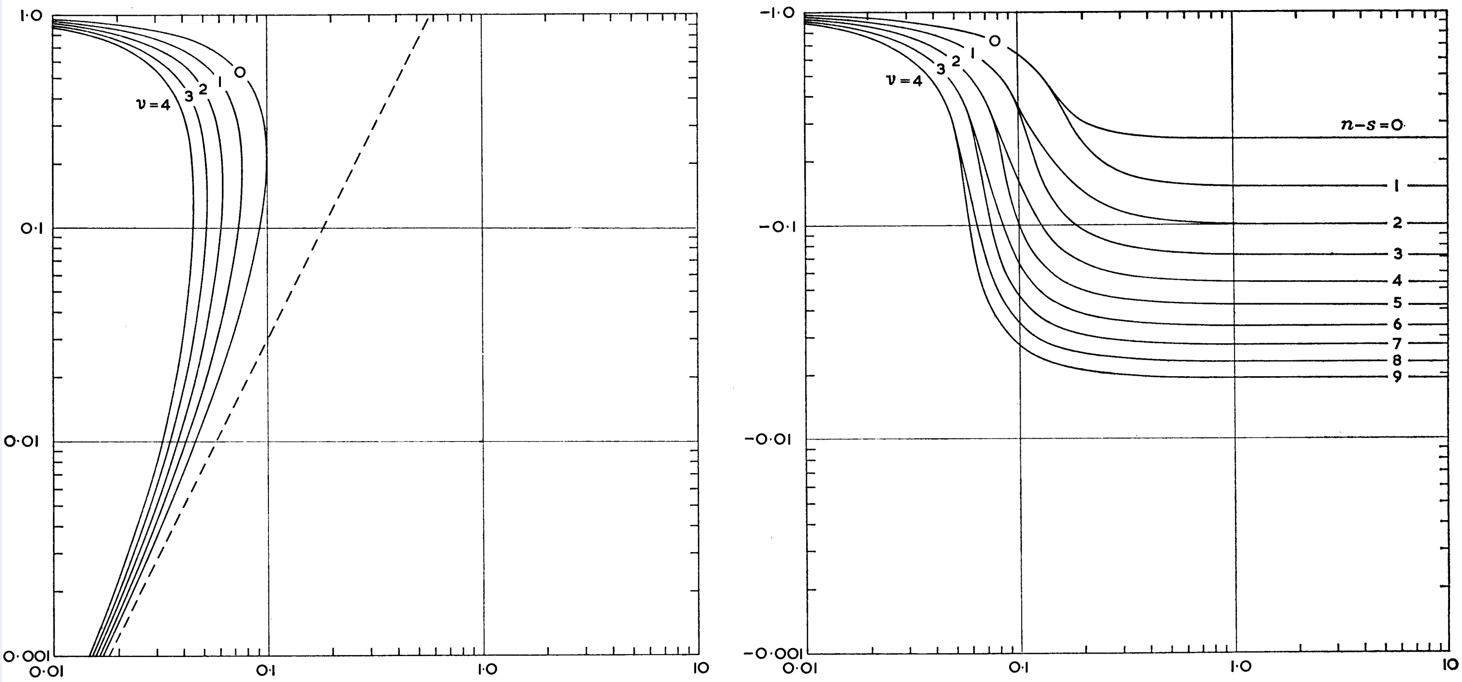


FIGURE 17. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 1$: (a) modes travelling eastwards, (b) modes travelling westwards.



(a) $\surd(-gh)/2\Omega$ (b)

FIGURE 18. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 2$: (a) modes travelling eastwards, (b) modes travelling westwards.



(a) $\surd(-gh)/2\Omega$ (b)

FIGURE 19. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 3$: (a) modes travelling eastwards, (b) modes travelling westwards.

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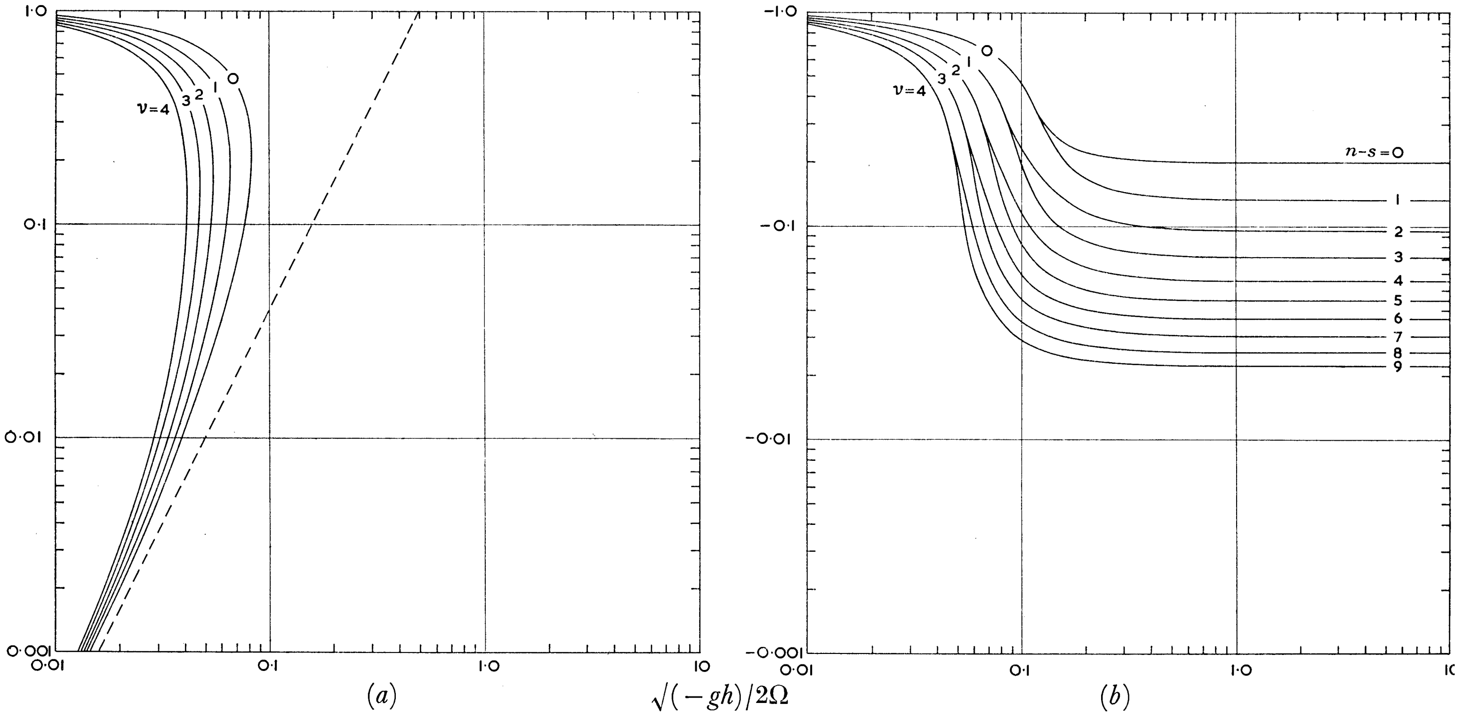

FIGURE 20. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 4$: (a) modes travelling eastwards, (b) modes travelling westwards.

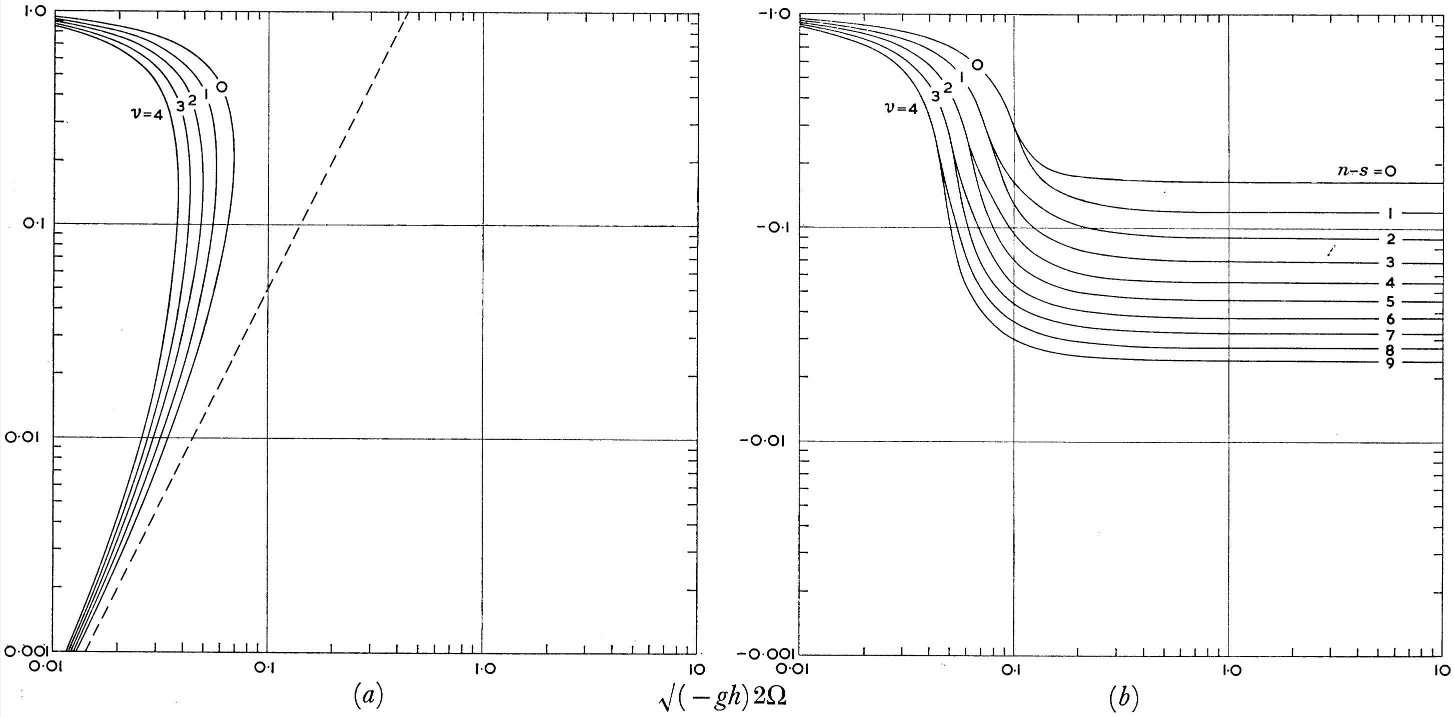

FIGURE 21. Eigenfrequencies of modes corresponding to negative values of ϵ, when $s = 5$: (a) modes travelling eastwards, (b) modes travelling westwards.

ω being of order unity, and assume also that

$$\lambda = -1 + \frac{Q}{(-\epsilon)^{\frac{1}{2}}} + O\left(\frac{1}{\epsilon}\right), \tag{11·3}$$

where Q is a constant to be determined. The last term in (11·1) is then approximately $(-\epsilon)^{\frac{1}{2}}(2Q-\omega^2)\,v^*$ which is of the appropriate order to balance the first term. Substituting in (11·1) and neglecting quantities of order $(-\epsilon)^{-\frac{1}{2}}$ we obtain

$$\left[\frac{d^2}{d\omega^2} - \frac{1}{\omega}\frac{d}{d\omega} - \frac{s^2-2s}{\omega^2} - \omega^2 + 2Q\right] v^* = 0. \tag{11·4}$$

The substitution

$$x = \omega^2 \tag{11·5}$$

reduces this to Whittaker's equation

$$\left[\frac{d^2}{dx^2} + \left(-\frac{1}{4} + \frac{k}{x} - \frac{4m^2-1}{x^2}\right)\right] v^* = 0, \tag{11·6}$$

where

$$k = \tfrac{1}{2}Q, \qquad 4m^2 - 1 = s^2 - 2s. \tag{11·7}$$

If m be assumed positive the last relation gives

$$2m = |s-1|. \tag{11·8}$$

The only solutions of (11·6) which are finite at both $x = 0$ and $x = \infty$ are given by

$$v^* \propto e^{-\frac{1}{2}x}\, x^{m+\frac{1}{2}}\, L_\nu^{(2m)}(x), \tag{11·9}$$

where $L_\nu^a(x)$ denotes the generalized Laguerre polynomial

$$L_\nu^a(x) \equiv \sum_{m=0}^{\infty} \binom{\nu+a}{\nu-m} \frac{(-x)^m}{m!}. \tag{11·10}$$

and ν is a non-negative integer related to k and m by

$$m + \tfrac{1}{2} - k = -\nu \quad (\nu = 0, 1, 2, 3, \ldots) \tag{11·11}$$

(see, for example, Erdelyi 1953, ch. 8). This last relation, with (11·7) implies that

$$Q = 2k = 2m + 2\nu + 1 = 2\,|s-1| + (2\nu+1). \tag{11·12}$$

So from (11·3) we have

$$\lambda = -1 + \frac{2\,|s-1| + (2\nu+1)}{(-\epsilon)^{\frac{1}{2}}}, \tag{11·13}$$

and from (11·9),

$$v^* \propto e^{-\frac{1}{2}\omega^2}\omega^{|s-1|+1} L_\nu^{|s-1|}(\omega^2). \tag{11·14}$$

At the south pole the solutions are exactly similar except that now $\omega = (-\epsilon)^{\frac{1}{4}}(\pi-\theta)$. The two halves of the solution are independent because at the equator they are both exponentially small.

This then is the explanation of the coalescence of the eigenvalues in pairs as $-\epsilon$ increases: each pair of eigenvalues represents two related solutions, one of which is symmetric and the other antisymmetric about the equator. For large values of $-\epsilon$ the only difference between

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the solutions is that in the symmetric mode the motion near the south pole is in phase with that near the north pole, and in the antisymmetric mode the motions are in antiphase. The frequencies are naturally almost equal.

Consider now the eastward-going waves. In the limiting case when $\lambda \doteqdot 1$ we may write

$$\lambda = 1 - \frac{Q}{(-\epsilon)^{\frac{1}{2}}} + O\left(\frac{1}{\epsilon}\right). \tag{11·15}$$

Proceeding as before we find that the analysis is identical except that s is replaced by $-s$. So replacing s by $-s$ in (11·12) we have for all values of $s \geqslant 0$

$$\lambda \doteqdot 1 - \frac{2s+2\nu+3}{(-\epsilon)^{\frac{1}{2}}}. \tag{11·16}$$

Finally consider the asymptotic form of the east-going modes in the other limiting case, namely when $-\epsilon\lambda \doteqdot s$. Assuming now that

$$\lambda = \frac{s}{-\epsilon} + \frac{2Qs}{(-\epsilon)^{\frac{3}{2}}} + O\left(\frac{1}{\epsilon^2}\right), \tag{11·17}$$

and with $\omega = (-\epsilon)^{\frac{1}{4}}\theta$ as before, we find that equation (11·1) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\omega^2} + \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega} - \frac{s^2}{\omega^2} - \omega^2 + 2Q\right] v^* = 0. \tag{11·18}$$

This can be written in the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\omega^2} - \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}w} - \frac{s^2-1}{\omega^2} - \omega^2 + 2Q\right] (\omega v^*) = 0, \tag{11·19}$$

which is identical with equation (11·4) except that $(s+1)$ replaces s and ωv^* replaces v^*. Accordingly we can write down the solution at once. Corresponding to (11·12) we have

$$Q = 2s+2\nu+1, \tag{11·20}$$

and so from (11·17)
$$\lambda \doteqdot \frac{s}{-\epsilon} + \frac{4s+4\nu+2}{(-\epsilon)^{\frac{3}{2}}}. \tag{11·21}$$

Corresponding to (11·14) we have

$$\omega v^* \propto \mathrm{e}^{-\frac{1}{2}\omega^2}\omega^{s+1}L_\nu^s(\omega^2), \tag{11·22}$$

and so
$$v^* \propto \mathrm{e}^{-\frac{1}{2}\omega^2}\omega^s L_\nu^s(\omega^2). \tag{11·23}$$

The asymptotic expressions for λ given by (11·16) and (11·21) are found to agree well with the computed values in figures 17 to 21. The appropriate value of ν has been assigned to each curve.

We see now that at both ends of the range of λ the energy is confined to the neighbourhood of the poles, so that the curves come together in pairs at each end of the range.

To summarize, we have found three new asymptotic forms of solution as $\epsilon \to -\infty$. These we shall denote by types 4 to 6 respectively.

Type 4

When $s \geqslant 1$ the eigenvalues, from (11·13), are given by

$$\lambda = -1 + \frac{2s+2\nu-1}{(-\epsilon)^{\frac{1}{2}}} \quad (\nu = 0, 1, 2, \ldots), \tag{11·24}$$

and v^*, from (11·14) is given by

$$v^* = e^{-\frac{1}{2}\omega^2}\omega^s L_\gamma^{(s-1)}(\omega^2)\, e^{i(s\phi-\sigma t)} \quad (\omega = (-\epsilon)^{\frac{1}{4}}\theta). \tag{11·25}$$

From equation (7·1) we have

$$v = -i(-\epsilon)^{\frac{1}{4}}\, e^{-\frac{1}{2}\omega^2}\omega^{s-1} L_\nu^{(s-1)}(\omega^2)\, e^{i(s\phi-\sigma t)}. \tag{11·26}$$

From (7·4)

$$\zeta^* = \frac{1}{(-\epsilon)^{\frac{1}{4}}}\left(\frac{1}{\omega}\frac{d}{d\omega} - \frac{s}{\omega^2}\right) v^*, \tag{11·27}$$

where v^* is given by (11·25). From (7·2) we have $u^* = -iv^*$ and so

$$u = -(-\epsilon)^{\frac{1}{4}}\, e^{-\frac{1}{2}\omega^2}\omega^{s-1} L_\nu^{(s-1)}(\omega^2)\, e^{i(s\phi-\sigma t)}. \tag{11·28}$$

Hence the motion takes place in inertial circles. From the expression for the total energy:†

$$-\tfrac{1}{4}\rho h \iint [|u^2| + |v^2| + \epsilon\,|\zeta^{*2}|]\, dS, \tag{11·29}$$

and equations (11·26) to (11·28) it is clear that the potential energy is small compared to the kinetic energy. Counting the energy in both hemispheres, the above expression becomes

$$\pi\rho h \int_0^\infty [|u^2| + |v^2|]\, \theta\, d\theta = 2\pi\rho h \int_0^\infty e^{-\omega^2}\omega^{2s-1}[L_\nu^{(s-1)}(\omega^2)]^2\, d\omega$$

$$= \frac{\pi\rho h}{\nu!\,(s-1)!\,[(\nu+s-1)!]^2}, \tag{11·30}$$

where we have used the result that

$$\int_1^\infty e^{-x}x^c\,[L_\nu^c(x)]^2\, dx = \frac{1}{\nu!\,c!\,[(\nu+c)!]^2}. \tag{11·31}$$

When $s = 0$ the motion is identical with that in type 5 (except for the immaterial change in sign of λ).

Type 5

When $s \geqslant 0$ we have from (11·16)

$$\lambda = 1 - \frac{2s+2\nu+3}{(-\epsilon)^{\frac{1}{2}}} \quad (\nu = 0, 1, 2, 3, \ldots) \tag{11·32}$$

and

$$v^* = e^{-\frac{1}{2}\omega^2}\omega^{s+2} L_\nu^{s+1}(\omega^2)\, e^{i(s\phi-\sigma t)}. \tag{11·33}$$

Hence we have

$$v = -i(-\epsilon)^{\frac{1}{4}}\, e^{-\frac{1}{2}\omega^2}\omega^{s+1} L_\nu^{(s+1)}(\omega^2)\, e^{i(s\phi-\sigma t)}. \tag{11·34}$$

† Both h and ϵ are taken to be negative by convention. The kinetic energy is positive, the potential energy negative.

From (7·4),

$$\zeta^* = -\frac{1}{(-\epsilon)^{\frac{1}{2}}}\left(\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega}+\frac{s}{\omega^2}\right)v^*, \tag{11·35}$$

and from (7·2) $u^* = \mathrm{i}v^*$ and so

$$u = (-\epsilon)^{\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}\omega^2}\omega^{s+1}L_\nu^{(s+1)}(\omega^2)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{11·36}$$

So the motion is in inertial circles. The energy is again almost totally kinetic and is given by

$$\pi\rho h\nu!\,(s+1)!. \tag{11·37}$$

The above formulae remain valid when $s = 0$.

Type 6

From (11·21) and (11·23) we have when $s \geqslant 1$

$$\lambda = \frac{s}{-\epsilon}+\frac{4s+4\nu+2}{(-\epsilon)^{\frac{3}{2}}} \quad (\nu = 0, 1, 2, 3, \ldots) \tag{11·38}$$

and

$$v^* = \mathrm{e}^{-\frac{1}{2}\omega^2}\omega^s L_\nu^{(s)}(\omega^2)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{11·39}$$

Hence

$$v = -\mathrm{i}(-\epsilon)^{\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}\omega^2}\omega^2 L_\nu^{(s)}(\omega^2)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{11·40}$$

From (7·4) and (11·38) we now have $\zeta^* = -v^*/s$, that is

$$\zeta^* = -\frac{1}{s}\,\mathrm{e}^{-\frac{1}{2}\omega^2}\omega^s L_\nu^{(s)}(\omega^2)\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}. \tag{11·41}$$

From the second of equations (7·2) we see that $u^* = -D\zeta^*$ and so

$$u = (-\epsilon)^{\frac{1}{4}}\frac{\mathrm{d}\zeta^*}{\mathrm{d}\omega}, \tag{11·42}$$

where ζ^* is given by (11·41). Hence the potential energy in this case greatly exceeds the kinetic energy. It is given by

$$-\tfrac{1}{4}\rho h\iint \epsilon|\zeta^{*2}|\,\mathrm{d}S = \pi\rho h\int_0^\infty (-\epsilon)^{\frac{1}{2}}\frac{1}{s^2}\,\mathrm{e}^{-\omega^2}\omega^{2s+1}[L_\nu^{(s)}(\omega^2)]^2\,\mathrm{d}\omega$$

$$= \frac{\pi\rho h(-\epsilon)^{\frac{1}{2}}}{2s^2\nu!\,s!\,[(\nu+s)!]^2}. \tag{11·43}$$

Type 7

When $s = 0$ we see from figure 16 that a still further limiting form arises, in which $\lambda \to 0$ for finite values of $-\epsilon$. Now when $s = 0$ equation (11·1) becomes

$$\left[\nabla^2+2\mu\frac{\mathrm{d}}{\mathrm{d}\mu}+\epsilon(\lambda^2-\mu^2)\right]v^* = 0, \tag{11·44}$$

that is
$$\left[(1-\mu^2)\frac{d^2}{d\mu^2}+\epsilon(\lambda^2-\mu^2)\right]v^* = 0. \tag{11·45}$$

When $\lambda \to 0$ this becomes
$$\left[(1-\mu^2)\frac{d^2}{d\mu^2}-\epsilon\mu^2\right]v^* = 0. \tag{11·46}$$

The limiting values of ϵ in figure 16 are accordingly the eigenvalues of this apparently simple equation. These limiting values, as determined by the present calculation, are shown below in table 10. The eigenvalues occur in pairs, each pair corresponding to one symmetric and one antisymmetric mode. The mth eigenvalue ϵ appears to increase in magnitude with m roughly like m^2.

12. The eigenfunctions for negative ϵ

We have seen that when ϵ is negative the integrals I_1 and I_2 are of opposite sign. Thus the quantity E, defined by (9·1), may actually vanish. For the purpose of normalizing the eigenfunctions we therefore define the quantity E' by

$$|I_1|+|I_2| = 4\pi E'. \tag{12·1}$$

Generally E' is not independent of the time, so we take its mean value $\bar{E}'$ and define the scales of velocity and surface elevation by

$$q_0 = |8\bar{E}'/\rho h|^{\frac{1}{2}}, \quad \zeta_0 = |8\bar{E}'/\rho g|^{\frac{1}{2}}. \tag{12·2}$$

Then u', v', ζ' and U, V, Z may be defined as in (9·3) and (9·6) respectively. This ensures that

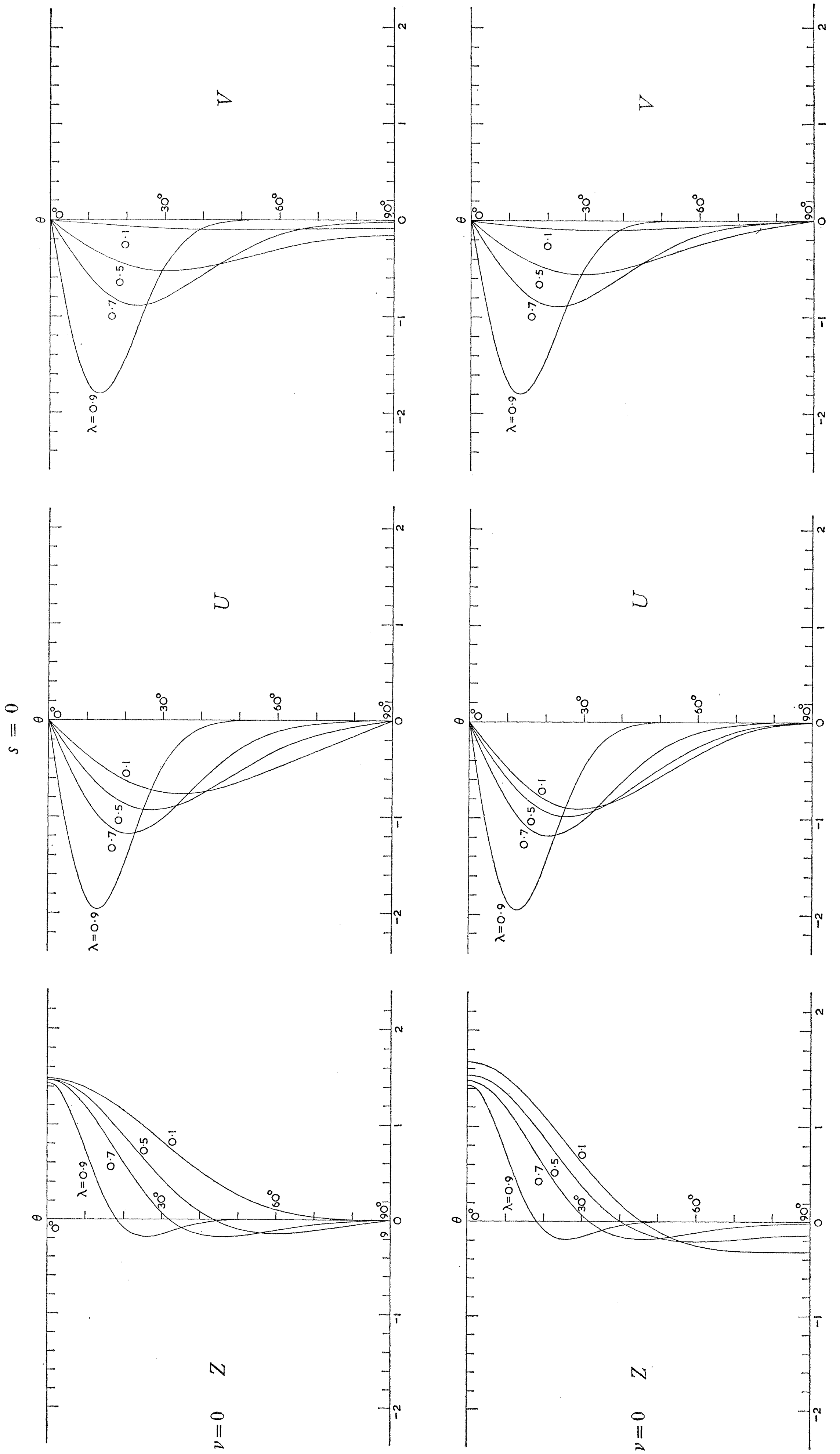
$$\int_{-1}^{1}(U^2+V^2+Z^2)\,d\mu = 1 \tag{12·3}$$

as before.

In figures 22 to 28 are shown the eigenfunctions Z, U and V, calculated at fixed values of the frequency σ, that is to say at fixed values of λ. This method of display has been chosen since the eigenfunctions are most likely to be required in solving a problem of *forced* motion, where the frequency is fixed rather than ϵ (see § 13). For the corresponding values of ϵ, see table 9.

Typical examples of the westwards mode ($\lambda < 0$) are shown in figures 25 and 28. It can be seen that as λ approaches -1 (for example when $\lambda = 0·9$) the energy tends to become concentrated near the pole, and that there is a vanishingly small disturbance near the equator, as was expected from the asymptotic formulae for waves of type 4.

Examples of the eastwards modes are shown in figures 23, 24, 26 and 27. In order to keep the diagrams clear these have been divided into two groups, corresponding to $1 > \lambda \geqslant 0·1$ and $0·1 > \lambda \geqslant 0·01$ respectively. Thus in figures 23 and 26 ($1 > \lambda \geqslant 0·1$) one can see that as λ approaches $+1$ the energy becomes concentrated near the poles as in the asymptotic type 5, and in figures 24 and 25 one can see the same phenomenon as λ approaches 0 (asymptotic type 6). However, even at intermediate values of λ, say $\lambda = 0·1$ in figures 23 and 26, the motion near the equator is always weak. There is very little difference between, say, the two lowest modes, except that one is symmetric and the other antisymmetric with respect to the equator.



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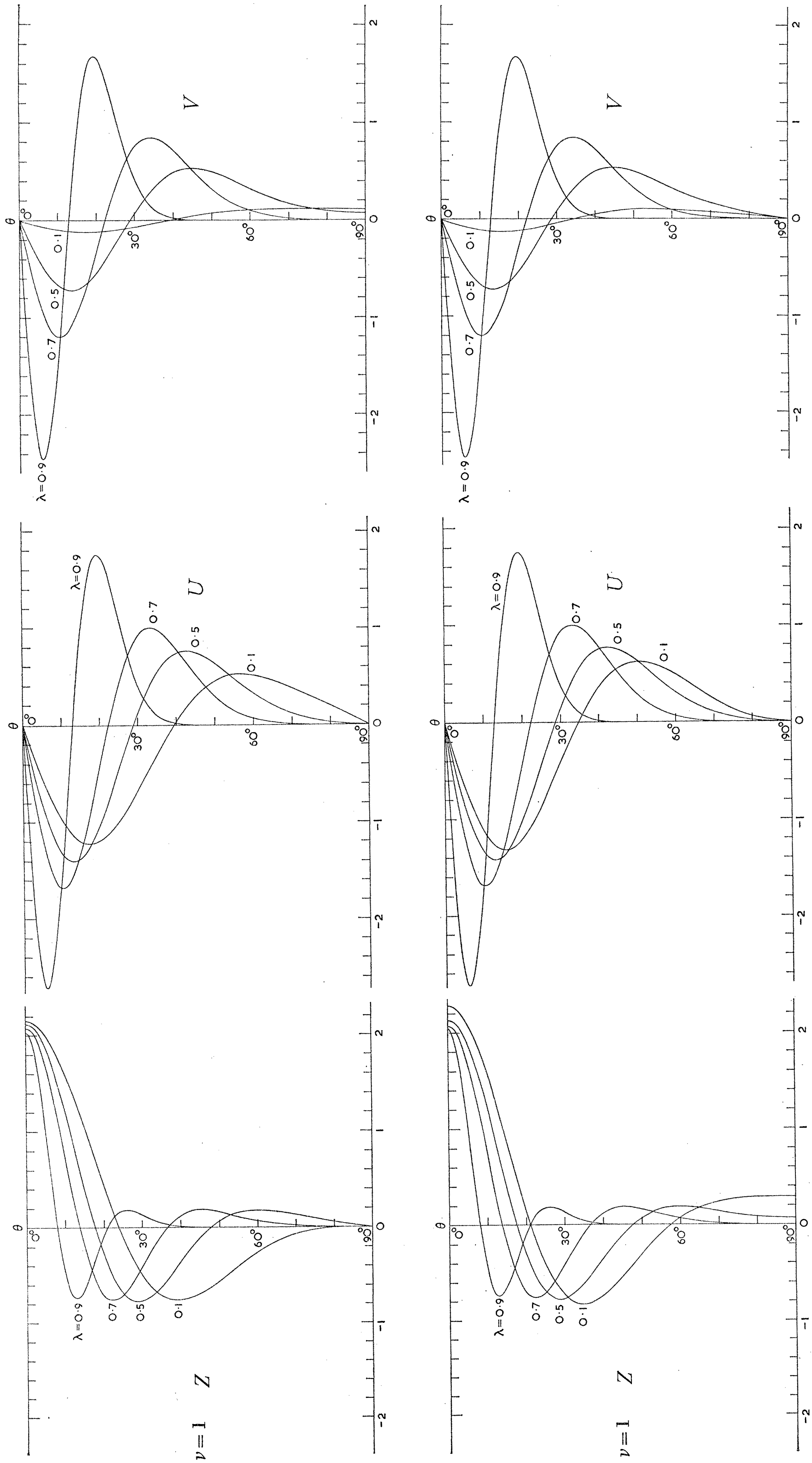
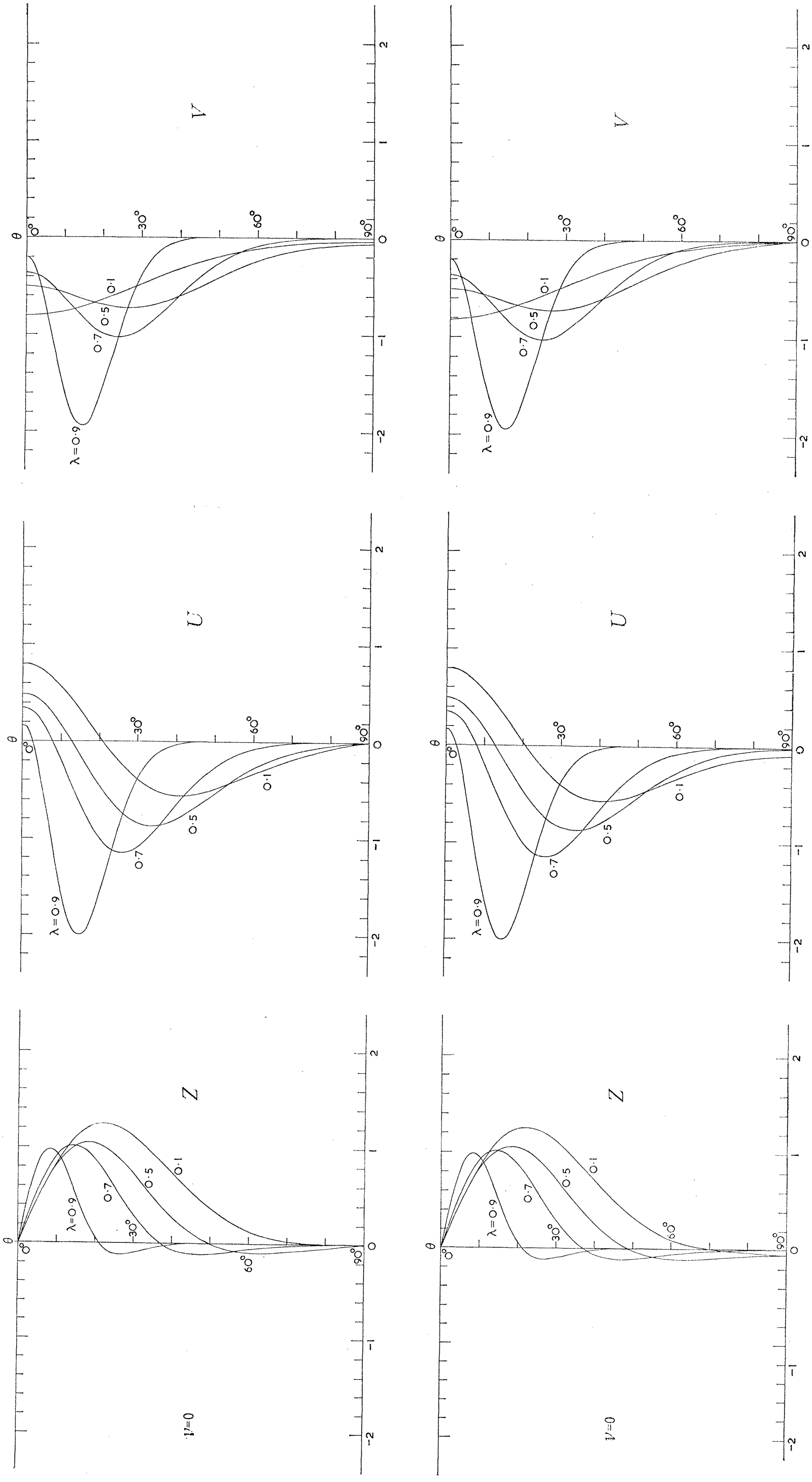


Figure 22. Eigenfunctions Z, U and V corresponding to negative values of ϵ, when $s = 0$. The four lowest modes.

eastward modes, $\epsilon < 0$, $s = 1$



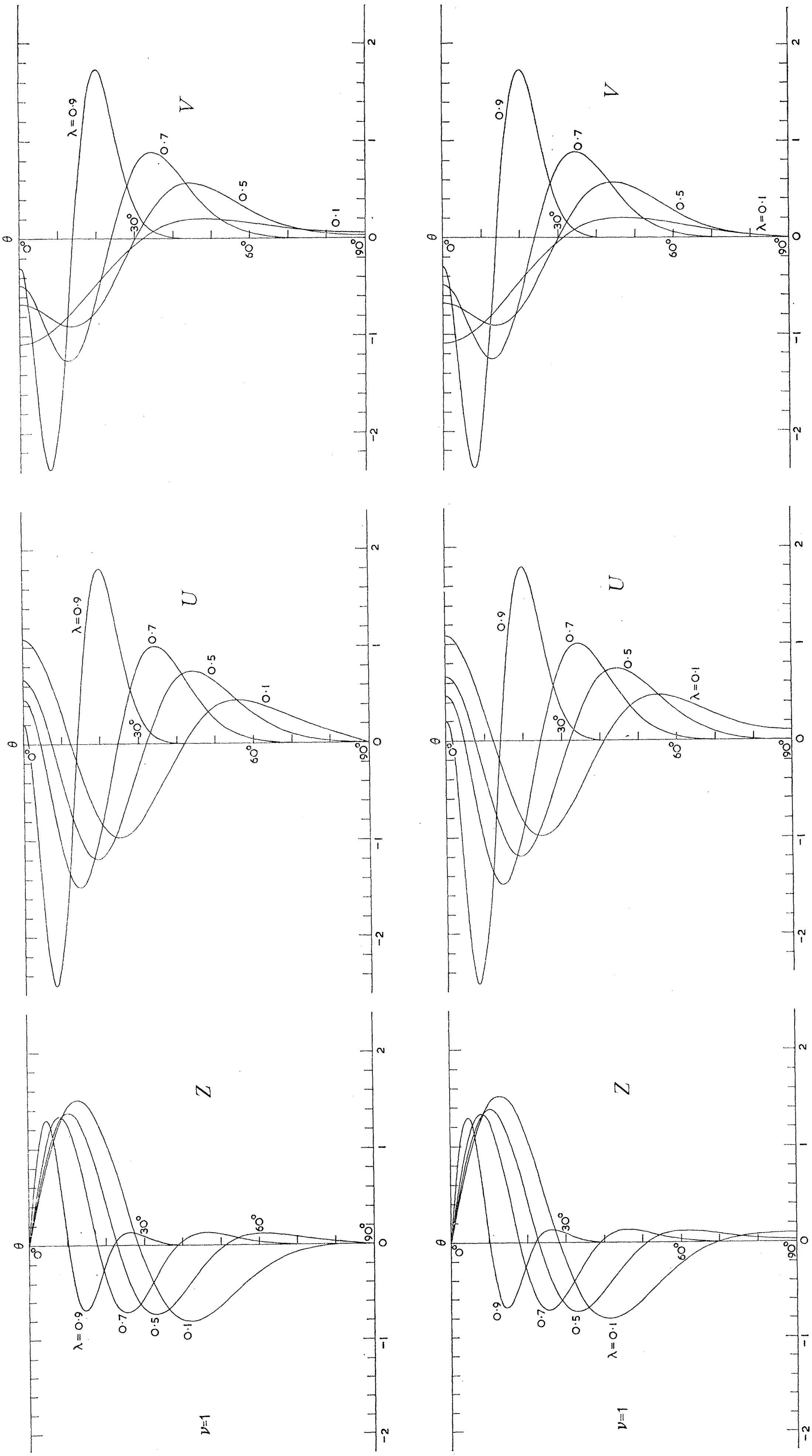
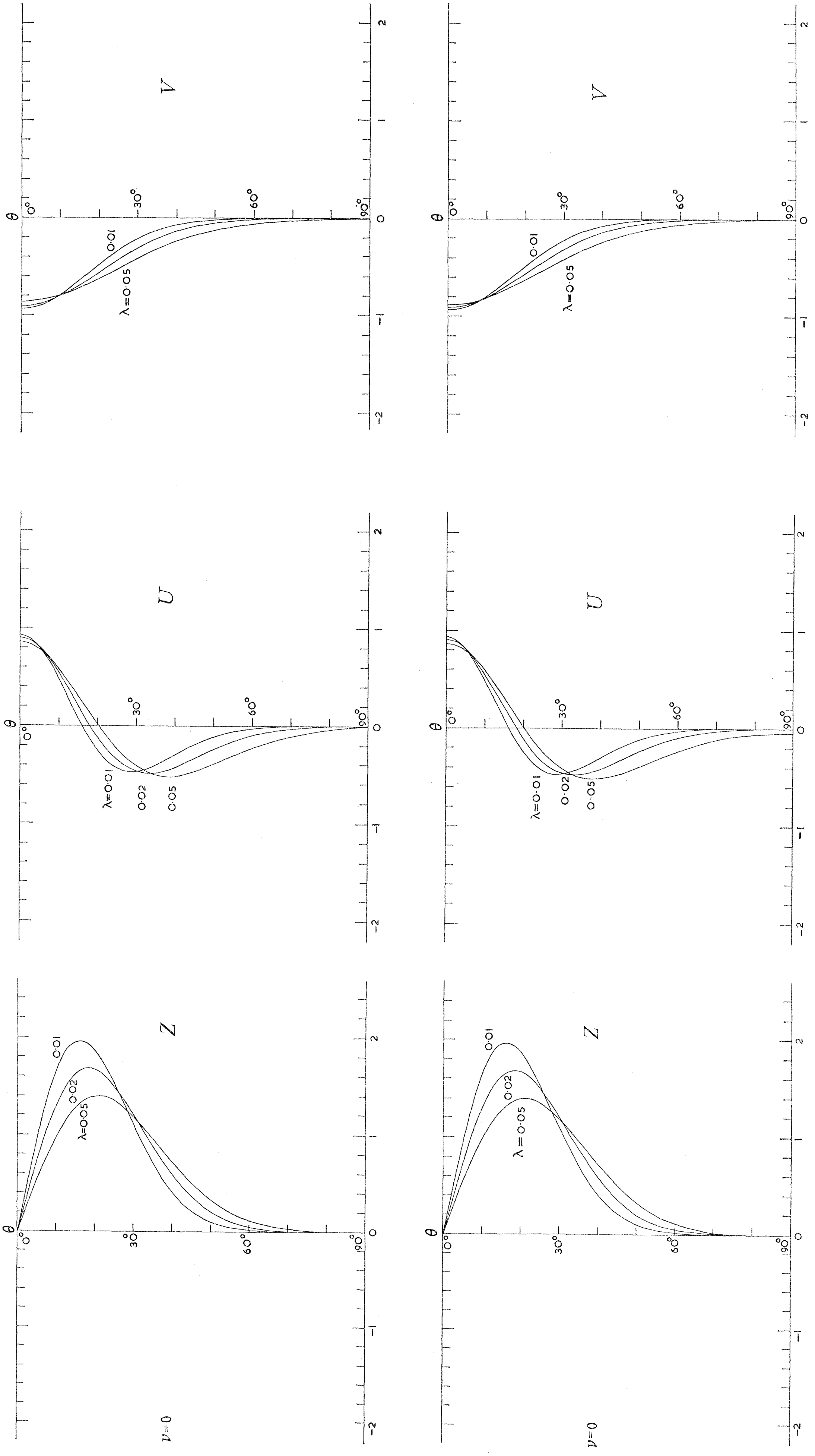


FIGURE 23. Eigenfunctions corresponding to negative values of ϵ when $s = 1$. Eastwards modes when $1 > \lambda \geqslant 0{\cdot}1$.

eastward modes, $\epsilon < 0$, $s = 1$ (*cont.*)



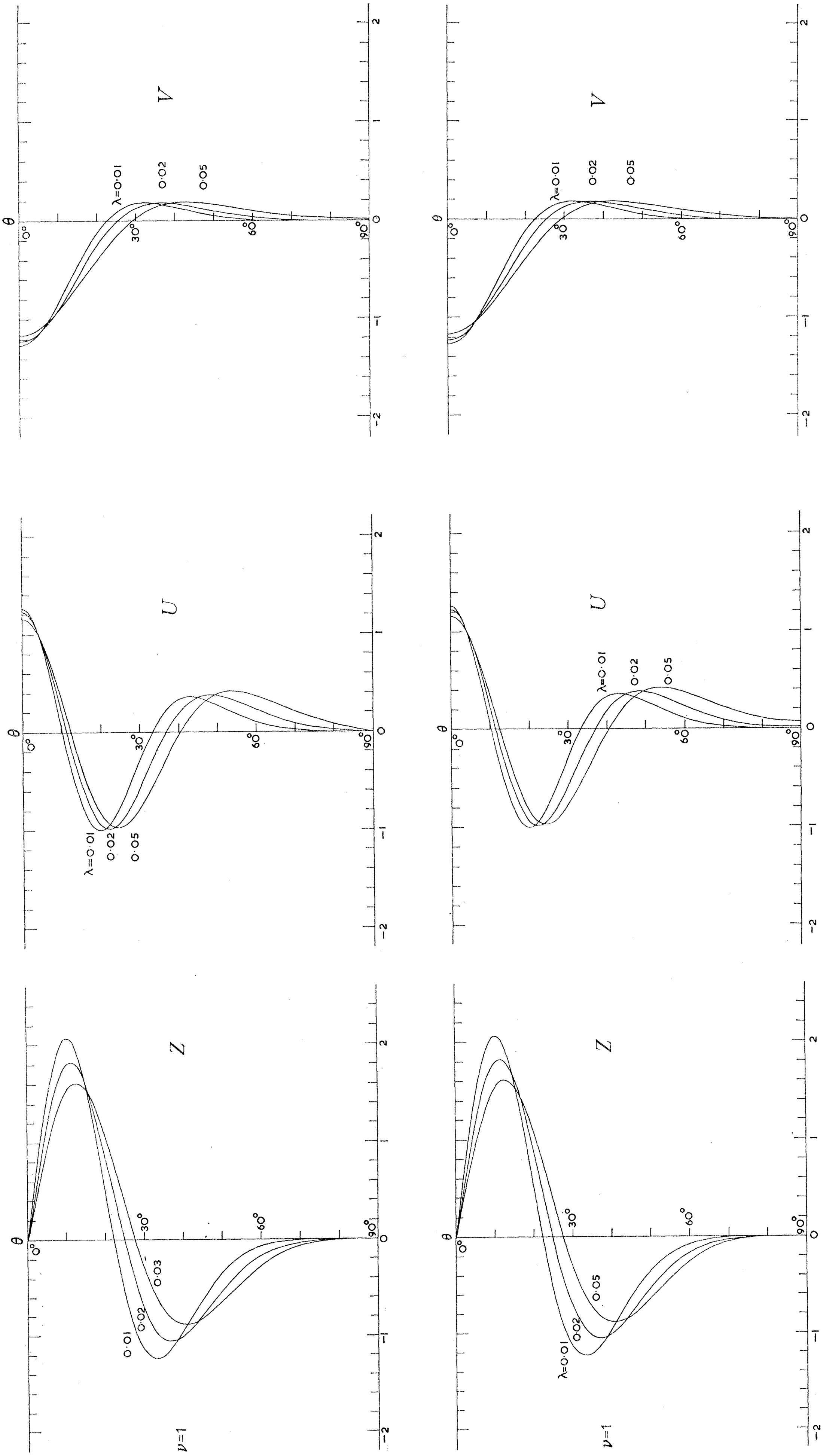


FIGURE 24. Eigenfunctions corresponding to negative values of ϵ when $s = 1$. Eastwards modes when $0{\cdot}1 > \lambda \geqslant 0{\cdot}01$.

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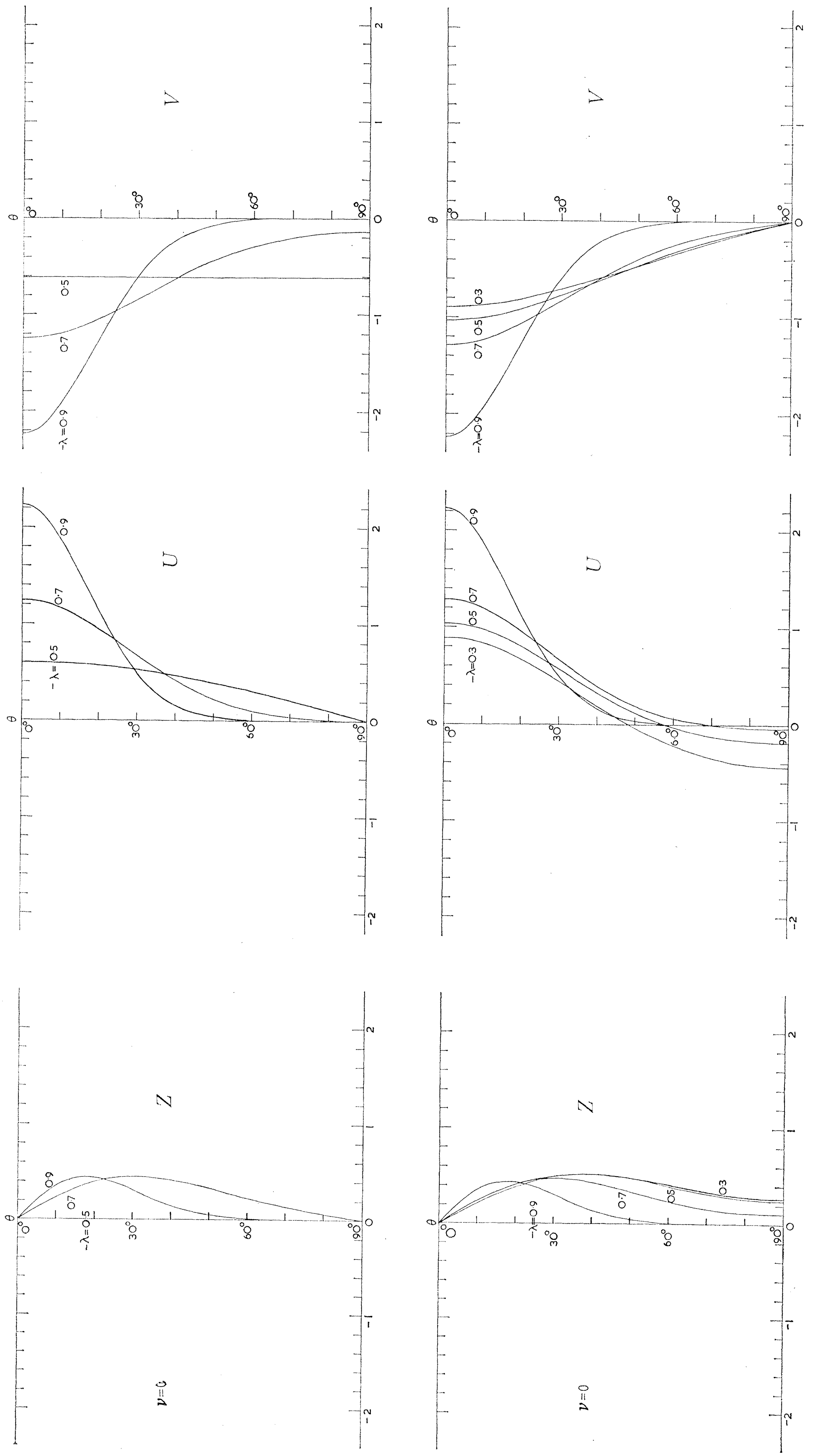
westward modes, $\epsilon < 0$, $s = 1$



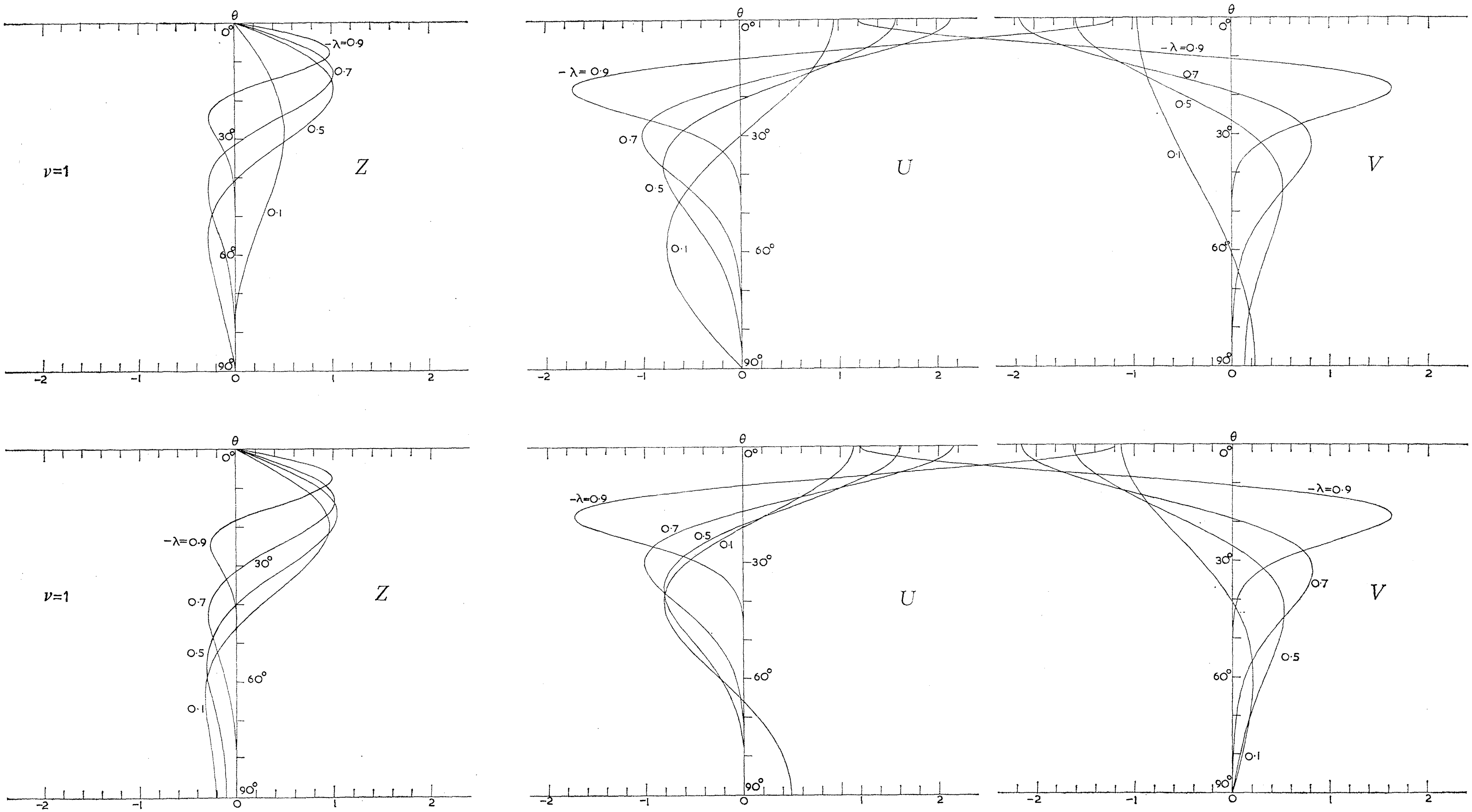
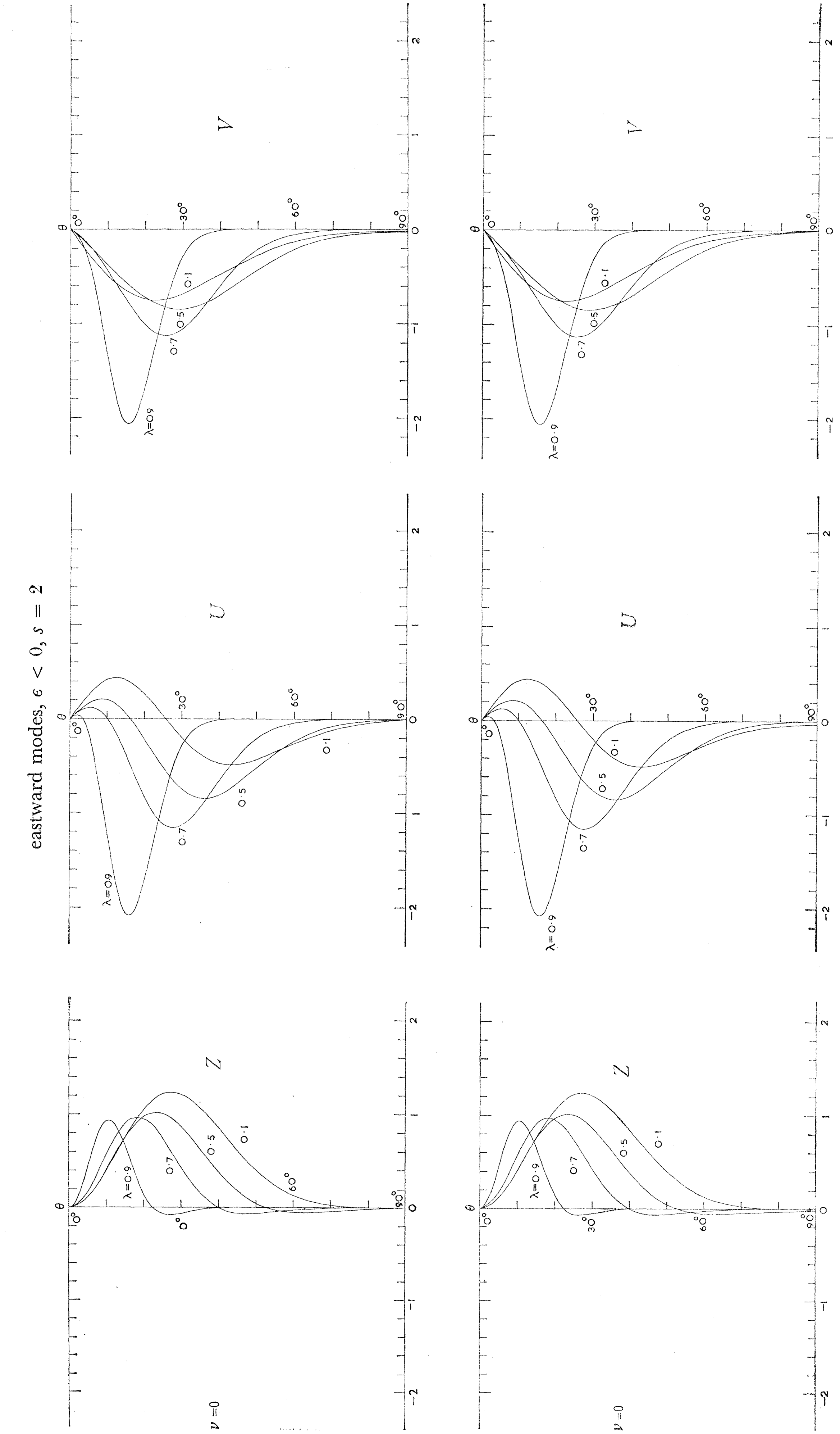


FIGURE 25. Eigenfunctions corresponding to negative values of ϵ when $s = 1$. Westwards modes.



eastward modes, $\epsilon < 0$, $s = 2$

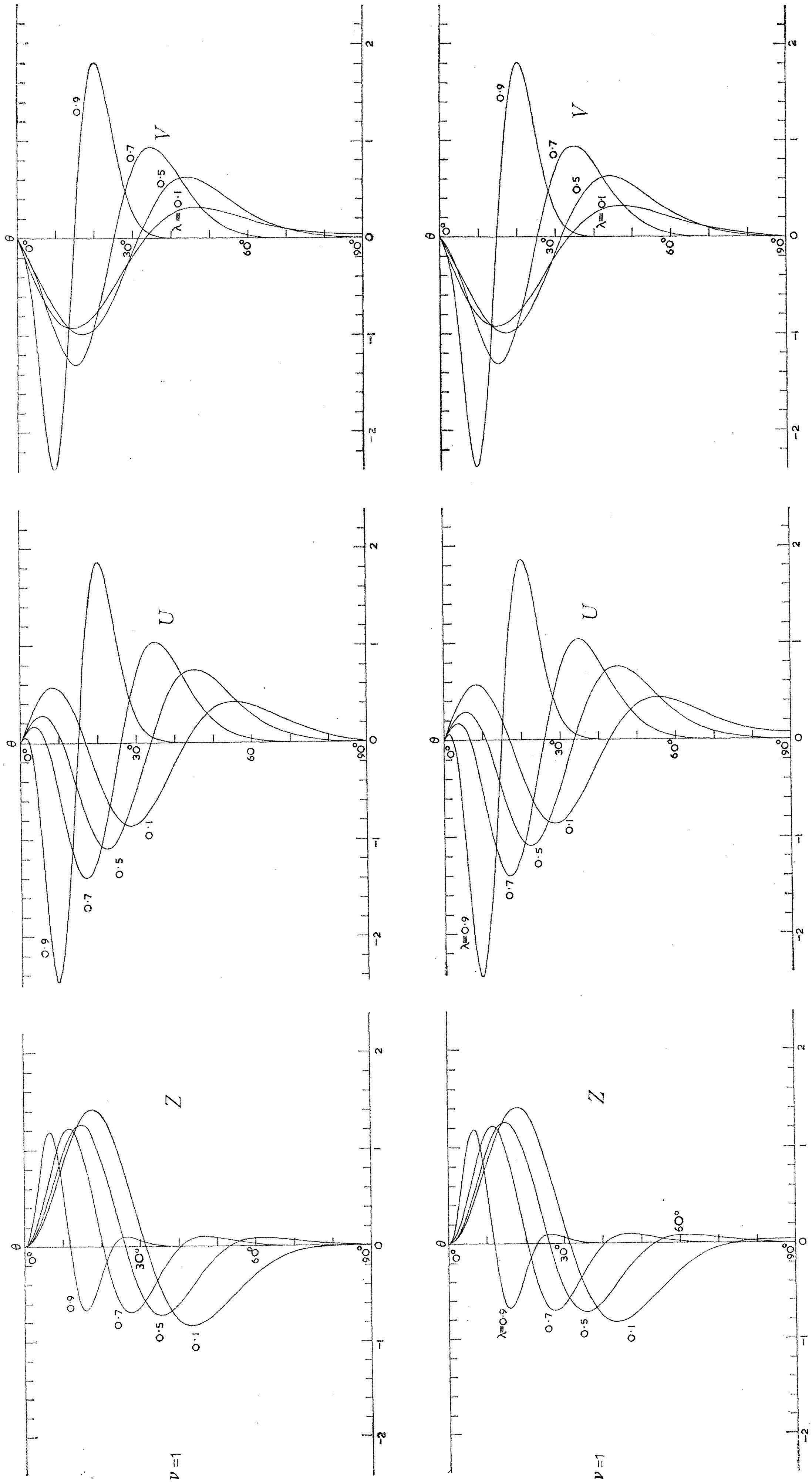
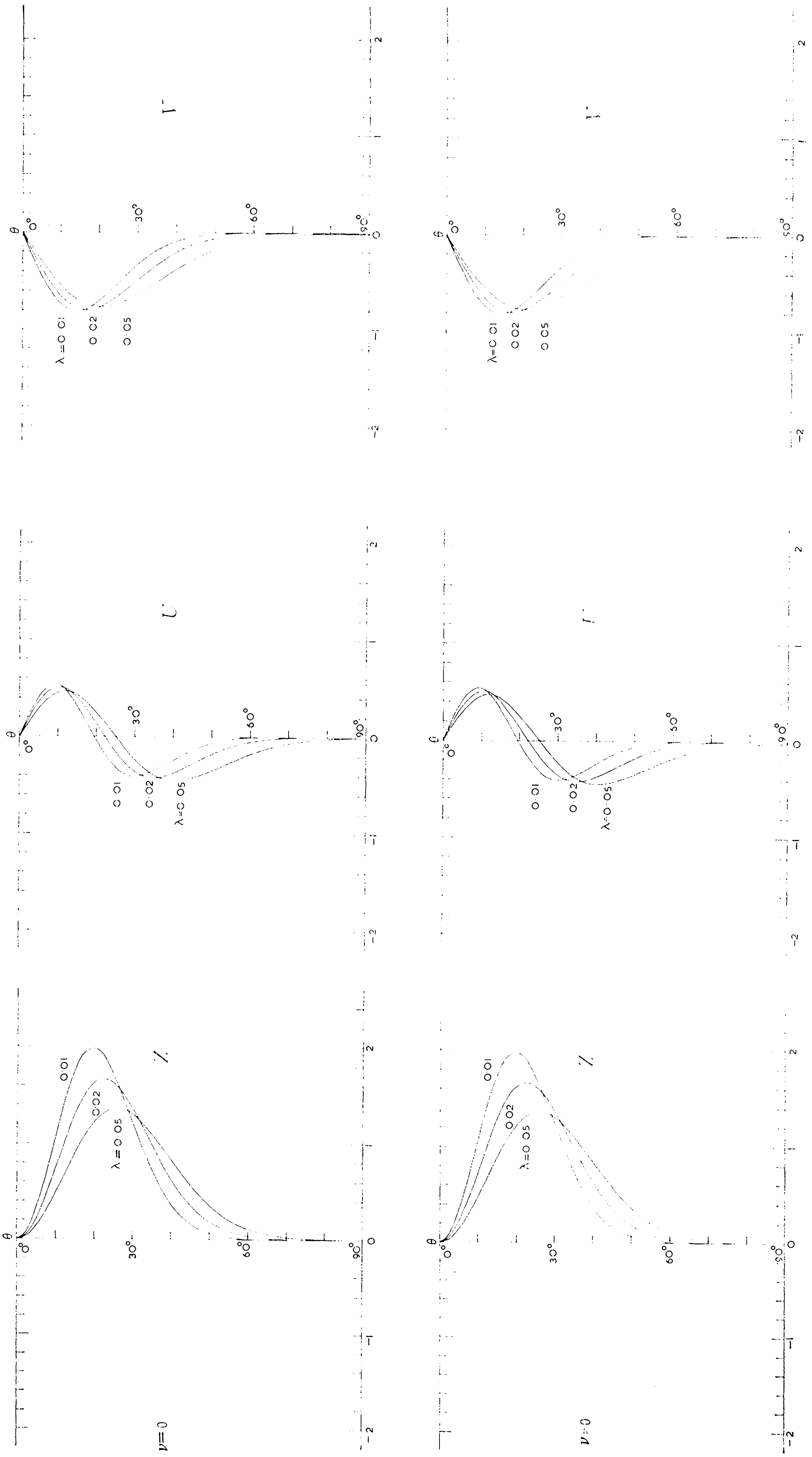


FIGURE 26. Eigenfunctions corresponding to negative values of ϵ when $s = 2$. Eastwards modes when $1{\cdot}0 > \lambda \geqslant 0{\cdot}1$.

eastward modes, $\epsilon < 0$, $s = 2$ (*cont.*)



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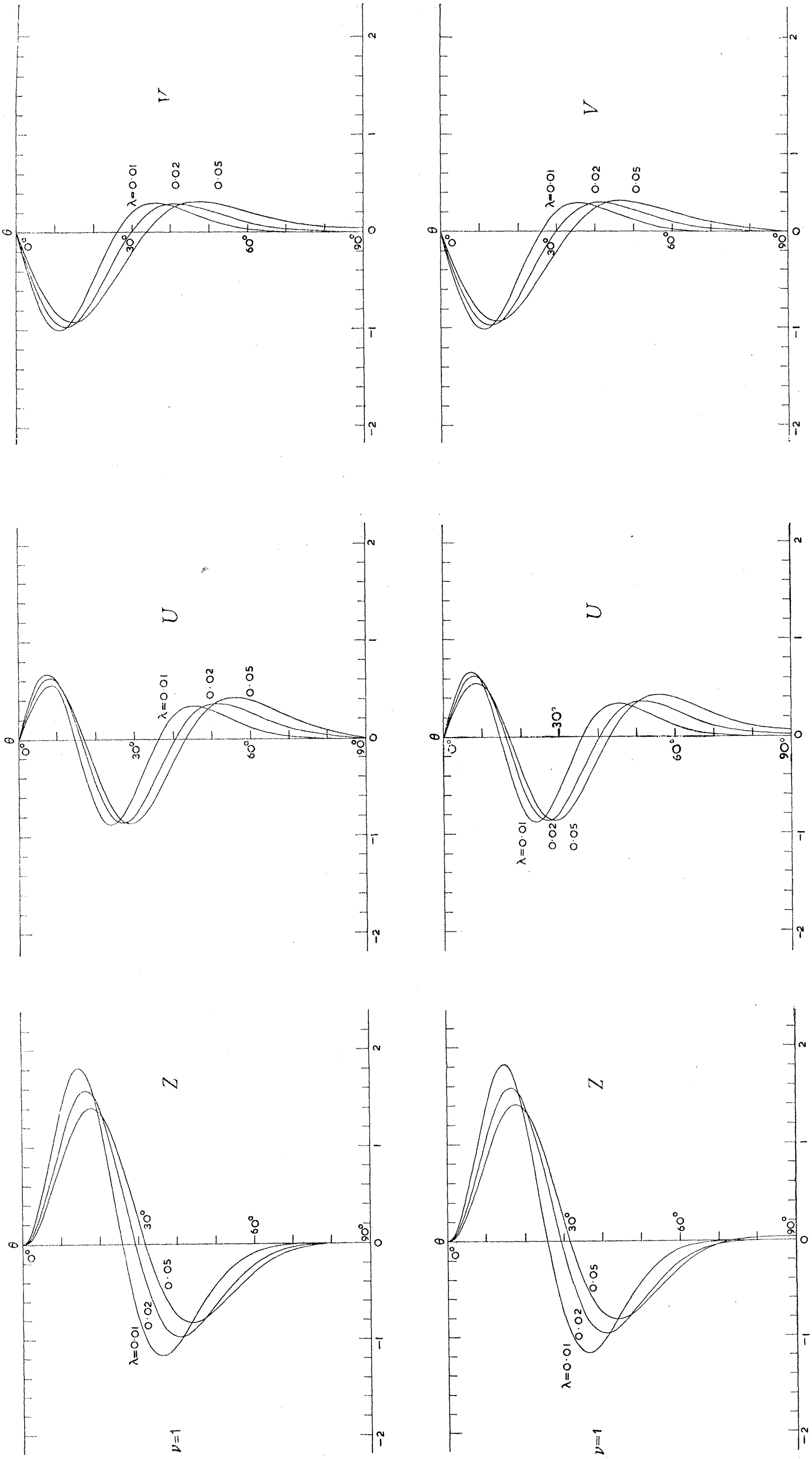
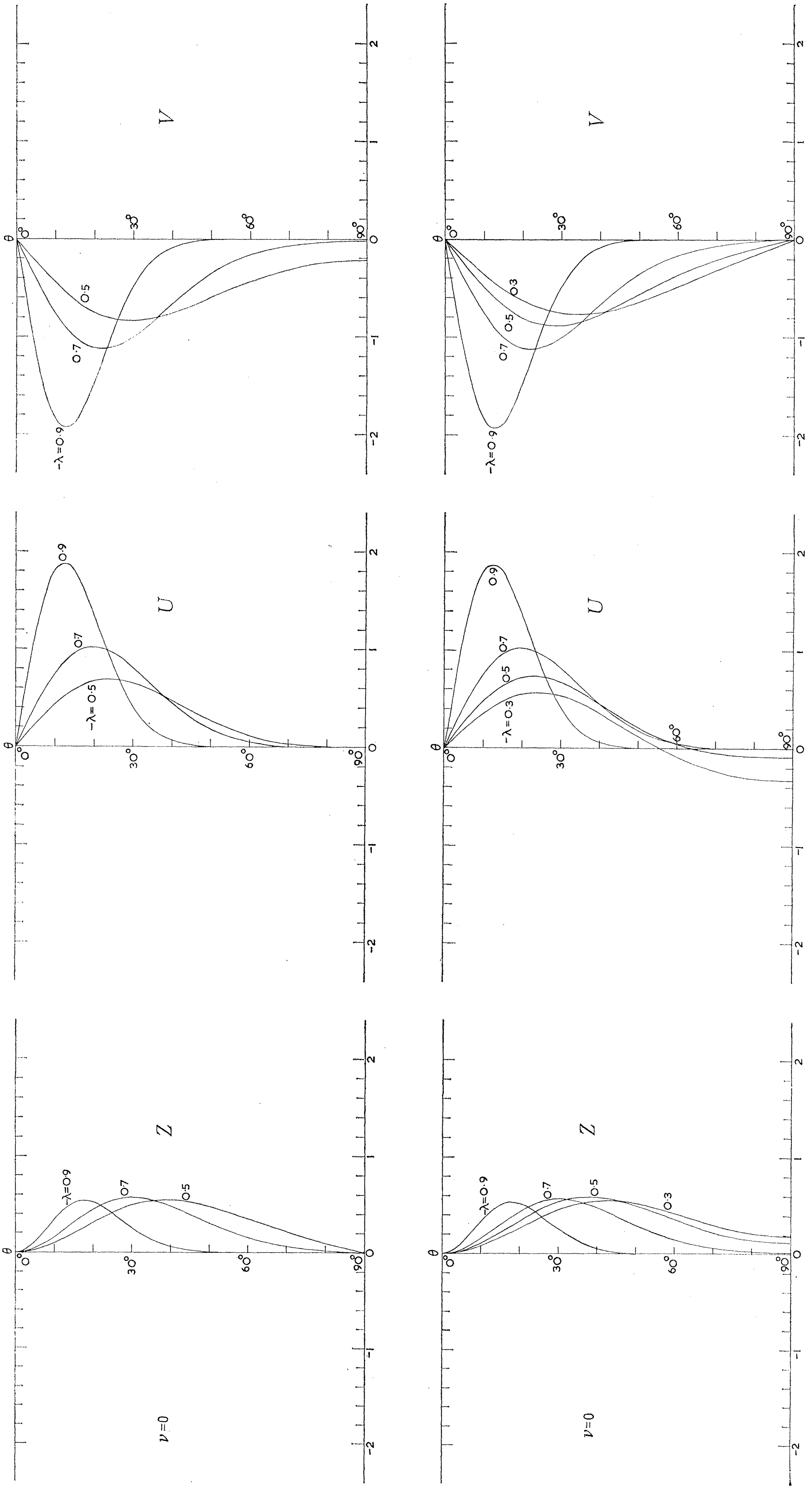


FIGURE 27. Eigenfunctions corresponding to negative values of ϵ when $s = 2$. Eastwards modes when $0{\cdot}1 > \lambda \geqslant 0{\cdot}01$.

westward modes, $\epsilon < 0$, $s = 2$



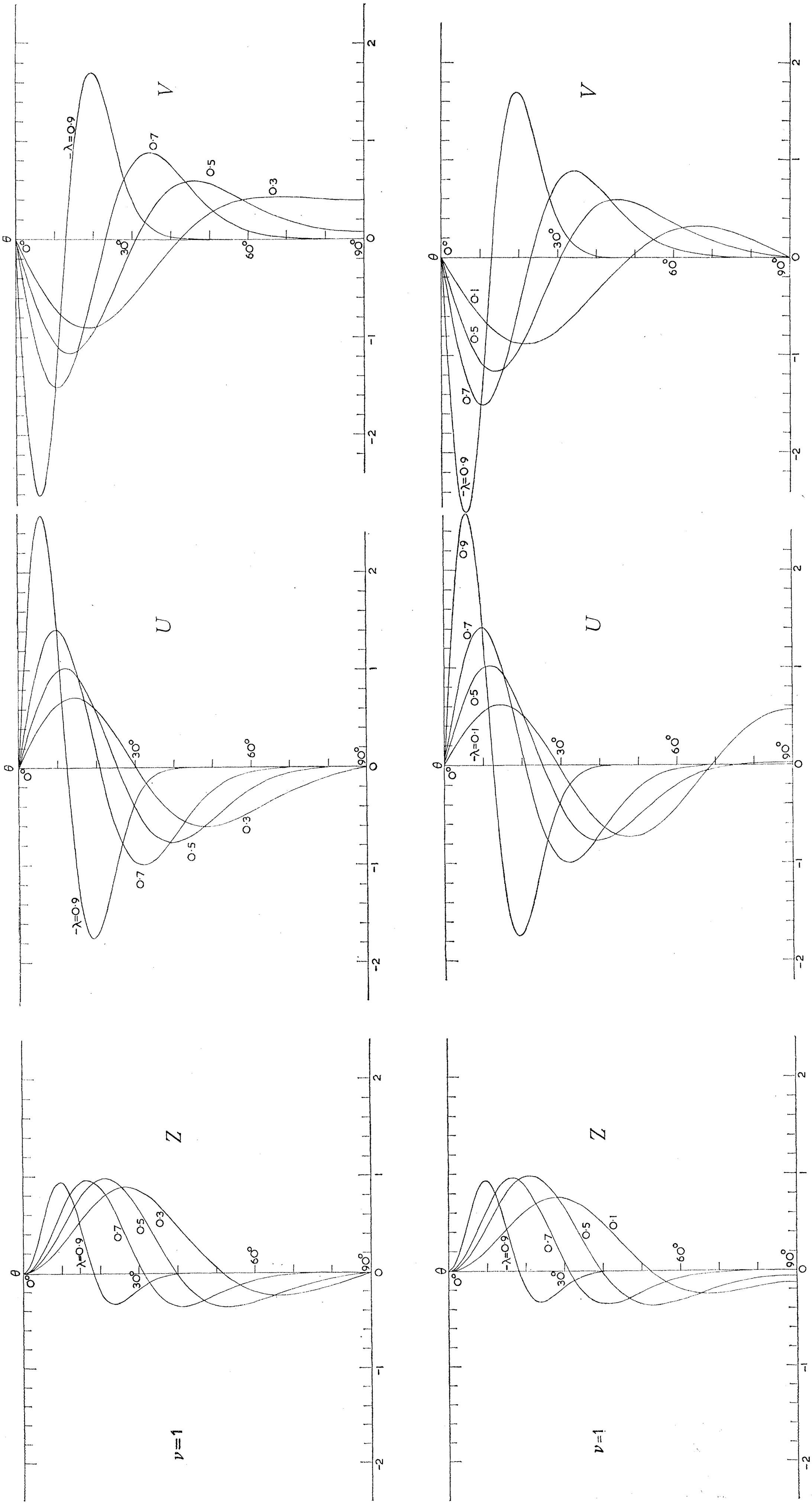


FIGURE 28. Eigenfunctions corresponding to negative values of ϵ when $s = 2$. Westwards modes.

The proportion of kinetic energy $|\bar{I}_1|$ in each mode relative to the sum $|\bar{I}_1|+|\bar{I}_2|$ has been plotted in figure 29 as a function of $(-\epsilon)^{-\frac{1}{2}}$. In the eastwards modes ($s = 1$ and 2) the proportion of kinetic energy varies from 1 to 0, and the curves nearly coincide in pairs throughout their entire length. On the other hand, in the westwards modes the proportion of kinetic energy has a minimum at the centre of the range and tends to unity at each end. At the right-hand end of the range, that is as $(-\epsilon)^{-\frac{1}{2}} \to \infty$ and $\epsilon \to 0$, the asymptotic behaviour of the curves is at first sight complicated. However, we may note that in this case the eigenfunctions tend to planetary waves (class 2) so that the formulae (4·10) apply for small negative ϵ as well as for small positive ϵ. Hence, using (9·12), we find

$$\frac{|\bar{I}_1|}{|\bar{I}_1|+|\bar{I}_2|} \doteqdot 1-\frac{|\epsilon|}{2n+1}\left[\frac{(n+1)\,(n^2-s^2)}{(2n-1)\,n^3}+\frac{n[(n+1)^2-s^2]}{(2n+3)\,(n+1)^3}\right]. \tag{12·4}$$

The coefficient of $-|\epsilon|$ is not quite monotonic in n, for given s. For $s = 1$ and 2 it takes the following values:

s \ $n-s$	0	1	2	3	4	5
1	0·02500	0·09194	0·04503	0·02636	0·01730	0·01224
2	0·01058	0·03009	0·02166	0·01536	0·01129	0·00860

The asymptotic value as $n\to\infty$ is $\frac{1}{2}n^{-2}$. Thus the coefficients are monotonic, with the exception of $(n-s) = 0$. This is in agreement with figure 29.

The proportion of kinetic energy in the westwards modes appears always to exceed $\frac{1}{2}$. However, when $s = 0$ (zonal oscillations) it appears from figure 29 that at the limiting values of $-\epsilon$ (for which $\lambda \to 0$) the proportion of kinetic energy tends to $\frac{1}{2}$ in the limit as $\lambda \to 0$. Thus these quasi-steady currents have an energy which is half kinetic and half potential.

13. Application to forced oscillations

For the sake of consistency we have so far interpreted the solutions corresponding to negative values of ϵ as though they were free oscillations with negative values of the depth h. Such a situation can hardly be realized physically.

On the other hand, as Lindzen (1966) has emphasized, the negative eigenvalues do play a physically meaningful role in describing the response of the system to external forces; in other words, they can represent real *forced* oscillations.

To give physical content to the solutions we may consider one of the simplest such situations, in which the layer of fluid is subject to a tide-raising gravitational potential, proportional to $e^{i(s\phi-\sigma t)}$, where σ is a given fixed frequency. Thus instead of the homogeneous system of equations (2·1) to (2·3) we consider the non-homogeneous system in which (2·1) and (2·2) are replaced by

$$\frac{\partial u}{\partial t}-2\Omega\cos\theta\, v+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(g\zeta) = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(g\bar{\zeta}), \tag{13·1}$$

$$\frac{\partial v}{\partial t}+2\Omega\cos\theta\, u-\frac{\partial}{\partial\theta}(g\zeta) = -\frac{\partial}{\partial\theta}(g\bar{\zeta}), \tag{13·2}$$

$$\frac{\partial\zeta}{\partial t}+\frac{h}{\sin\theta}\left[\frac{\partial}{\partial\theta}(-v\sin\theta)+\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}\right] = 0, \tag{13·3}$$

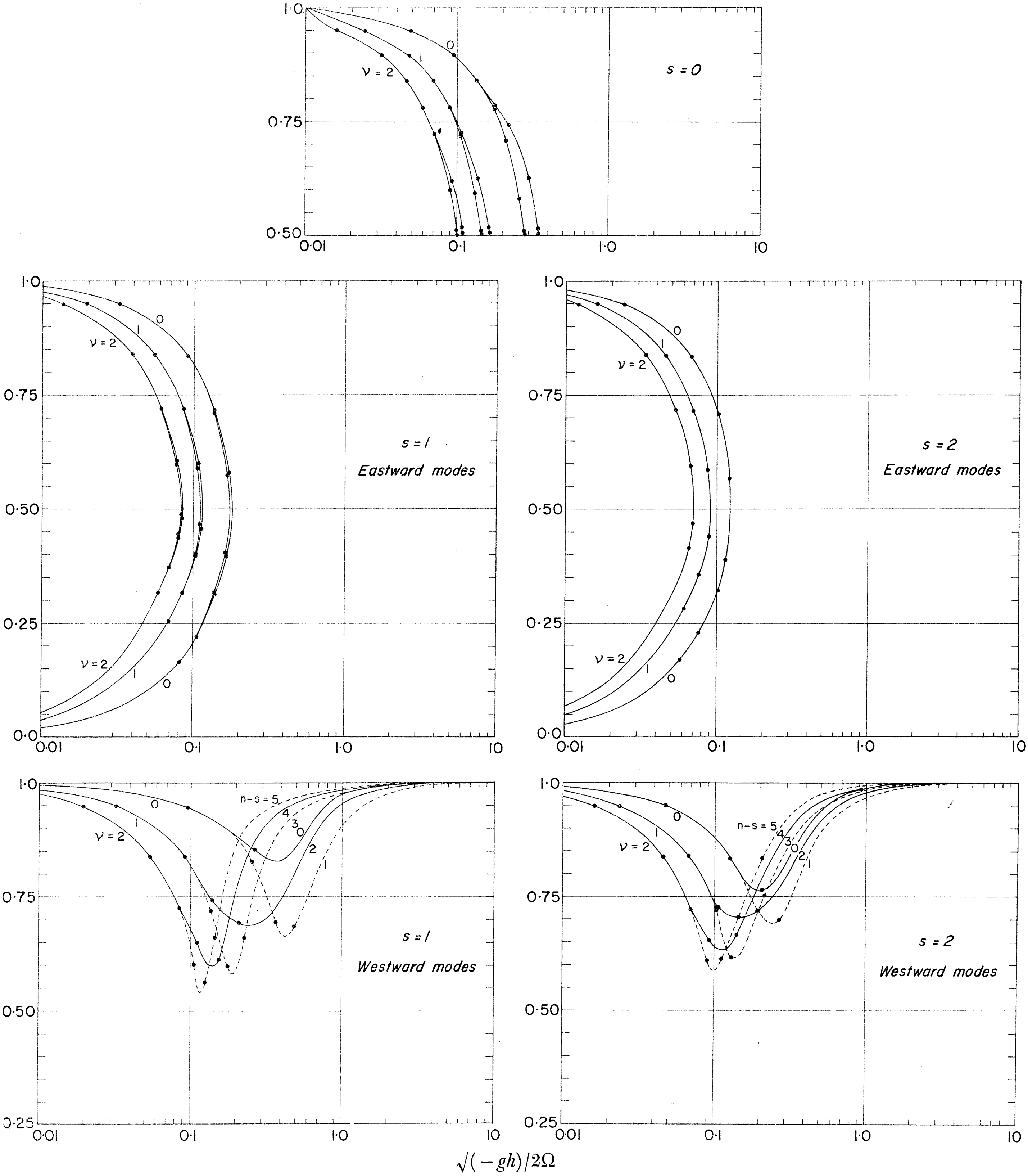


FIGURE 29. Proportion of kinetic energy to total energy in the modes corresponding to negative ϵ when $s = 0$, 1, and 2.

where $\bar{\zeta}$ represents an equilibrium tide. On taking $\bar{\zeta}$ over to the left-hand side of equations (13·1) and (13·2) we see that u and v can be expressed in terms of $(\zeta-\bar{\zeta})$ as in equations (2·7) but with $(\zeta-\bar{\zeta})$ replacing ζ. Hence we have in place of (2·9) the equation

$$\mathscr{L}(\zeta-\bar{\zeta}) = \lambda\epsilon\zeta \tag{13·4}$$

Suppose now that $\bar{\zeta}$ happens to be an eigenfunction of equation (2·9), that is to say

$$\bar{\zeta} = AZ_m\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{13·5}$$

where A is a constant, and

$$\mathscr{L}(Z_m) = \lambda\epsilon_m Z_m \tag{13·6}$$

for some ϵ_m (positive or negative). Then clearly (13·4) has the solution

$$\zeta = B\mathscr{L}_m\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{13·7}$$

where

$$\lambda\epsilon_m(B-A) = \lambda\epsilon B, \tag{13·8}$$

that is to say

$$B = \frac{\epsilon_m}{\epsilon_m-\epsilon}A. \tag{13·9}$$

In this equation ϵ stands for $4\Omega^2/gh$, which is positive in a real situation. If ϵ_m is positive, then $(\epsilon_m-\epsilon)$ may vanish, though for given λ and ϵ this is improbable. In that case we should have a resonant excitation of the mode Z_m. On the other hand, if ϵ_m is negative, $(\epsilon_m-\epsilon)$ cannot vanish. In other words, resonant excitation is impossible.

In the most general case of a periodic forcing function, $\bar{\zeta}$ may be expanded in a series of eigenfunctions of equation (2·9), that is to say we can write

$$\bar{\zeta} = \sum_m A_m Z_m\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)}, \tag{13·10}$$

where the A_m are constants. The solution of equation (13·3) can then be written down. It is

$$\zeta = \sum_m \frac{\epsilon_m A_m}{\epsilon_m-\epsilon} Z_m\,\mathrm{e}^{\mathrm{i}(s\phi-\sigma t)} \tag{13·11}$$

(assuming of course that $\epsilon_m \neq \epsilon$, for any m). Moreover, the general theory of equations such as (2·9) shows that for the complete representation of a general forcing function $\bar{\zeta}$ in the form (13·5) all of the eigenfunctions are necessary. In other words, those eigenfunctions corresponding to eigenvalues $\epsilon_m > 0$ would not by themselves form a complete set. The eigenfunctions with $\epsilon_m \leqslant 0$ therefore play an essential role in the solution to the problem.

14. Conclusions and summary

A direct numerical calculation of the eigenvalues and eigenfunctions of Laplace's equations (2·1) to (2·3) over a complete range of the parameter $\epsilon = 4\Omega^2/gh$ has revealed a wealth of asymptotic forms for the free oscillations of fluid on a rotating globe.

As $\epsilon \to 0$ through positive values, the asymptotic forms are the well-known gravity waves (waves of the first class) on the one hand or the planetary waves (waves of the second class) on the other.

As $\epsilon \to +\infty$ there are three types of waves (described analytically in § 8). In all three types

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the energy is trapped within an angular distance $O(\epsilon^{-\frac{1}{4}})$ from the equator. In the first type the frequency is asymptotically proportional to $\epsilon^{-\frac{1}{4}}$, and the kinetic energy exceeds the potential energy by a factor of 3. In the second and third types the frequency is proportional to $\epsilon^{-\frac{1}{2}}$, and the energy is equally divided between kinetic and potential. The waves of the second type are all propagated towards the west, those of the third type are 'Kelvin waves' propagated eastwards along the equator.

The transition from the asymptotic forms as $\epsilon \rightarrow 0$ to the forms as $\epsilon \rightarrow \infty$ is illustrated in figures 1 to 6. For any given positive value of the wave-number s, all but one of the waves of the first class as $\epsilon \rightarrow 0$ become waves of the first type as $\epsilon \rightarrow \infty$. The exceptional wave becomes a Kelvin wave (type 3). Similarly, all but one of the waves of the second class as $\epsilon \rightarrow 0$ become waves of the second type as $\epsilon \rightarrow \infty$. The exceptional wave becomes of type 1. In the case $s = 0$ the only waves (apart from zonal currents having zero frequency) are of class 1 as $\epsilon \rightarrow 0$, and these all become of type 1 as $\epsilon \rightarrow \infty$. Some of the above results have been found independently by Golitsyn & Dikii (1966).

When $\epsilon \rightarrow 0$ through negative values, there is only one asymptotic form of solution. These represent motions which are analytically continuous with the planetary waves (ϵ small and negative).

When $\epsilon \rightarrow -\infty$ there are again three types of solution (types 4, 5 and 6 of § 11) in all of which the energy is concentrated within a distance $O(-\epsilon)^{-\frac{1}{4}}$ from the *poles* of rotation. In types 4 and 5 the frequencies approach the inertial frequency at the poles and the energy is predominantly kinetic. The motion is in inertial circles. In type 6 (which is valid for positive s) the frequency is proportional to ϵ^{-1}, and the energy is predominantly gravitational potential energy. A consequence of the isolation of energy at the poles is that the modes tend to occur in pairs having nearly the same frequency. One member of each pair is symmetric with respect to the equator and the other is antisymmetric.

The transition of the modes between the various asymptotic forms is illustrated in figures 16 to 21. As ϵ goes from 0 to $-\infty$, the waves of class 2 go over into waves of type 4. The proportion of kinetic energy in these modes has a minimum in this range of ϵ. At small values of ϵ there are no waves corresponding to types 5 and 6 as $\epsilon \rightarrow -\infty$; the latter exist only when $\epsilon\lambda < -s$. Those waves which are of type 5 as $\epsilon\lambda \rightarrow -\infty$ become of type 6 when $\epsilon\lambda \rightarrow -s$.

Lastly, when $s = 0$ the waves of types 4 and 5 (which are then identical) go over into yet another asymptotic form (type 7) in which the frequency tends to zero for a finite (negative) value of ϵ. In the limit the kinetic and potential energies are equal.

The normalized eigenfunctions corresponding to positive values of ϵ are shown in figures 7 to 13, and those corresponding to negative values of ϵ are shown in figures 22 to 28. The number of zeros of each mode is not necessarily preserved throughout the range of ϵ for which the mode exists.

Although the eigenfunctions which correspond to negative values of ϵ do not represent physically real free motions, nevertheless a knowledge of these eigenfunctions is necessary to a complete solution of the problem of the forced motions induced by an arbitrary external field of force.

One feature may be mentioned which is common to all but one of the asymptotic solutions as $\epsilon \rightarrow \pm\infty$: the simplest dependent variable analytically is not ζ but $v^* = iv\sin\theta$.

This is proportional to the component of the particle velocity parallel to the axis of rotation. The exceptional case is type 3, or Kelvin, waves, where ζ is simpler than v^*.

The types of oscillation revealed by this investigation may have close analogues in other situations involving rotating fluids. One such example, that of the oscillations in a shallow rotating dish of paraboloidal shape, is described briefly in the Appendix.

We may also expect to find similar oscillations in closed regions of uniform or variable depth on the surface of a sphere. For example in a sector of the sphere bounded by meridians of longitude we expect to find Kelvin waves travelling eastwards along the equator and trapped there by Coriolis forces. Such waves cannot travel westwards. The continuity of energy flux is therefore probably maintained by a continuation of the Kelvin wave along the meridianal boundaries of the basin, as in figure 30. This possibility is being investigated by analytical methods.

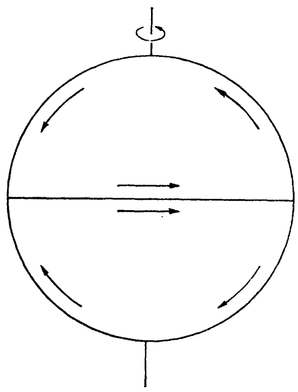

FIGURE 30. A sketch of the probable flux of energy in Kelvin waves, in an ocean bounded by two meridians of longitude.

APPENDIX. WAVES IN A ROTATING PARABOLIC BASIN

It was pointed out to the author by Professor L. N. Howard that the eigenfrequencies for waves in a rotating parabolic basin show some of the peculiar features that have been noted in the present paper for waves on a rotating sphere. Lamb (1932, para. 212) considers the problem of a shallow rotating dish whose mean depth h is given by

$$h = h_0(1-r^2/a^2), \tag{A 1}$$

where r is the radial horizontal coordinate and h_0 and a are constants signifying the maximum depth and the radius of the basin. He shows that the free modes have a surface elevation of the form

$$\zeta = (r/a)^s F(s+\nu+1, -\nu; s+1; r^2/a^2)\, e^{i(s\phi-\sigma t)}, \tag{A 2}$$

where ϕ is the angular coordinate in the horizontal plane, F denotes the hypergeometric series and

$$\nu = 0, 1, 2, 3, \ldots. \tag{A 3}$$

The corresponding frequency σ is found from the relation

$$\frac{(\sigma^2-4\Omega^2)\,a^2}{gh_0}+\frac{4\Omega s}{\sigma} = (s+2\nu+2)\,(s+2\nu)-s^2. \tag{A 4}$$

On writing $$\sigma/2\Omega = \lambda, \qquad 4\Omega^2 a^2/gh_0 = \epsilon \tag{A 5}$$

this becomes $$\epsilon\lambda^3 - [\epsilon + 2s(2\nu+1) + 4\nu(\nu+1)]\,\lambda + 2s = 0, \tag{A 6}$$

a simple cubic equation in λ, with ϵ as a parameter. In the special case $\nu = 0$ the root $\lambda = 1$ must be excluded.

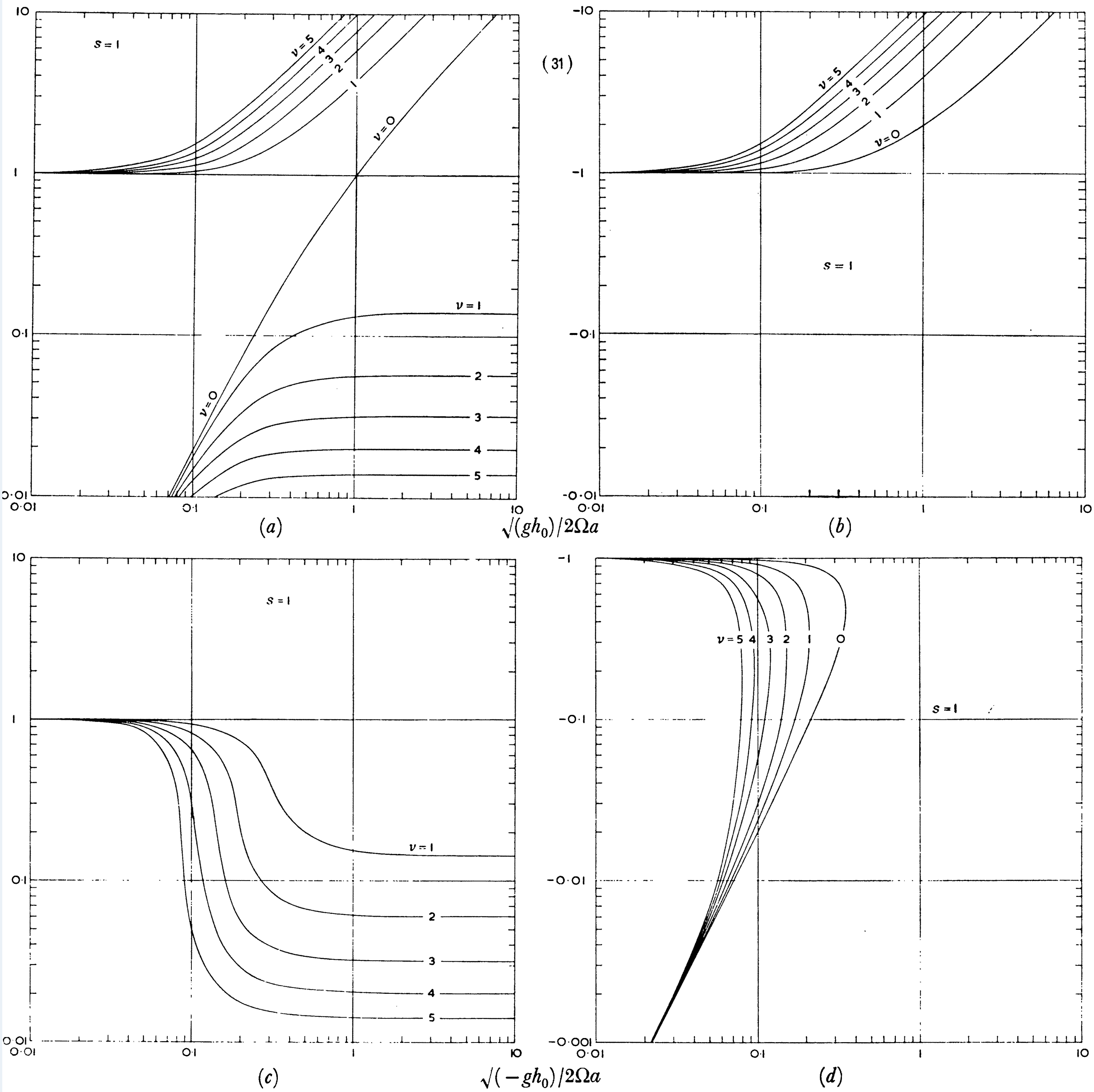

FIGURE 31. Eigenfrequencies of free modes of oscillation in a shallow paraboloidal dish of radius a and maximum depth h_0, rotating with angular velocity Ω.

The behaviour of the solutions of this cubic in the typical case when $s = 1$ is shown in figures 31 (*a*) to (*d*). Figures 31 (*a*) and (*b*) correspond to positive depths. At small rates of rotation there are, at most values of the parameters ϵ and ν, three possible waves, two gravity waves and one wave of the second class. One of the gravity waves travels towards the east and one towards the west. However, the wave of the second class travels always towards the *east*, the β-effect being reversed since the depth is greatest at the centre of the basin.

At high rates of rotation there are again two classes of waves. In one of these the frequency λ tends to ± 1; in the other λ is asymptotically proportional to ϵ^{-1}. However, one mode corresponding to $\nu = 0$ crosses over from the second group at high rates of rotation to the first group at low rates of rotation.

Figures 31 (*c*) and (*d*) show the eigenfrequencies corresponding to negative depths. As can be seen, there is a marked resemblance to the corresponding modes on the sphere.

The numerical calculations described in this paper were carried out on an I.B.M. 7094 digital computer at the I.B.M. Data Centre in London, and on a C.D.C. 3600 at the University of California, San Diego. I am indebted to both the National Institute of Oceanography and to the University of California for financial support for the computations.

In plotting the graphs and preparing the tables I have been ably assisted by Miss P. S. Gribble. The curves were traced by Mr A. Style.

For the reference to Lindzen (1966) I am indebted to Professor N. A. Phillips. The analogy described in the Appendix was suggested to me by Professor L. N. Howard.

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LIST OF TABLES

TABLE 1. EIGENVALUES λ when $\epsilon > 0$ and $s = 0$, FOR GIVEN VALUES OF η, $= 1/\epsilon\lambda$

$\log_{\sqrt{2}}\lvert\eta\rvert$	$n-s = 1$	2	3	4	5	6	7	8
9	±45·2593	±135·768	±271·531	±452·549	±678·823	±950·352	±1267·14	±1629·17
8	32·0063	96·0045	192·002	320·002	480·001	672·001	896·001	1152·00
7	22·6363	67·8886	135·768	226·276	339·413	475·177	633·568	814·588
6	16·0125	48·0089	96·0049	160·003	240·002	336·001	448·001	576·001
5	11·3314	33·9538	67·8891	113·141	169·708	237·590	316·785	407·295
4	8·02490	24·0178	48·0097	80·0060	120·004	168·003	224·002	288·002
3	5·69194	16·9958	33·9549	56·5770	84·8585	118·798	158·395	203·649
2	4·04925	12·0356	24·0194	40·0120	60·0081	84·0058	112·004	144·003
1	2·89708	8·53545	16·9980	28·3012	42·4379	59·4052	79·2022	101·828
0	2·09446	6·07049	12·0388	20·0240	30·0162	42·0117	56·0088	72·0069
−1	1·54132	4·34107	8·53995	14·1760	21·2362	29·7150	39·6104	50·9214
−2	1·16521	3·13588	6·07687	10·0479	15·0324	21·0233	28·0176	36·0137
−3	·912062	2·30518	4·35011	7·13845	10·6524	14·8822	19·8239	25·4753
−4	·740956	1·74107	3·14870	5·09450	7·56447	10·5466	14·0351	18·0274
−5	·621946	1·36338	2·32332	3·66706	5·39380	7·49023	9·94910	12·7667
−6	·534670	1·11131	1·76643	2·68054	3·87616	5·34208	7·06985	9·05468
−7	·466492	·939683	1·39772	2·00992	2·82533	3·84064	5·04769	6·44089
−8	·410320	·816812	1·15541	1·56385	2·10909	2·80158	3·63626	4·60768
−9	·362473	·721995	·992608	1·27296	1·63152	2·09399	2·66181	3·33132
−10	·321005	·643127	·875825	1·08341	1·32057	1·62286	2·00062	2·45379
−11	·284736	·574289	·783296	·954130	1·12060	1·31700	1·56247	1·86199
−12	·252841	·513014	·703302	·855997	·988022	1·12186	1·27964	1·47279
−13	·224696	·458175	·631307	·771971	·889731	·994196	1·10048	1·22363
−14	·199798	·409052	·565987	·695513	·805309	·899477	·983432	1·06692
−15	·177737	·365065	·506811	·625288	·727412	·816335	·893962	·962902
−16	·158163	·325702	·453358	·561080	·655132	·738412	·812471	·878235
−17	·140780	·290505	·405204	·502695	·588613	·665618	·735179	·798160
−18	·125332	·259053	·361923	·449842	·527862	·598391	·662790	·721897
−19	·111596	·230963	·323094	·402167	·472699	·536858	·595882	·650552
−20	·099377	·205888	·288309	·359279	·422830	·480906	·534622	·584691
−21	—	·18351	·25718	·32078	·37790	·43027	—	—

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Table 2. Eigenvalues λ when $\epsilon > 0$ and $s = 1, 2, \ldots, 5$.

Modes travelling eastwards

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s = 1$				
9	44·7582	135·600	271·447	452·499	678·790	950·328	1267·12	1629·16
8	31·5047	95·8370	191·919	319·951	479·968	671·977	895·983	1151·99
7	22·1341	67·7208	135·684	226·226	339·379	475·153	633·551	814·574
6	15·5096	47·8407	95·9211	159·953	239·969	335·978	447·983	575·987
5	10·8274	33·7848	67·8052	113·091	169·675	237·566	316·768	407·281
4	7·51950	23·8480	47·9255	79·9557	119·971	167·979	223·984	287·988
3	5·18485	16·8246	33·8703	56·5266	84·8250	118·774	158·377	203·635
2	3·54036	11·8625	23·9344	39·9614	59·9745	83·9819	111·987	143·990
1	2·38685	8·35958	16·9123	28·2504	42·4042	59·3812	79·1842	101·814
0	1·58471	5·89072	11·9520	19·9728	29·9824	41·9876	55·9908	71·9929
−1	1·03619	4·15587	8·45178	14·1243	21·2020	29·6908	39·5924	50·9074
−2	·671475	2·94333	5·98665	9·99537	14·9980	20·9990	27·9994	35·9997
−3	·436995	2·10334	4·25701	7·08494	10·6175	14·8576	19·8056	25·4611
−4	·289186	1·52861	3·05156	5·03954	7·52892	10·5216	14·0167	18·0132
−5	·195169	1·14034	2·22063	3·61005	5·35735	7·46483	9·93034	12·7523
−6	·133760	·879222	1·65632	2·62066	3·83844	5·31603	7·05072	9·04002
−7	·0926201	·701172	1·27830	1·94596	2·78583	3·81368	5·02805	6·42592
−8	·0645658	·575377	1·02534	1·49397	2·06707	2·77336	3·61591	4·59228
−9	·0452100	·482123	·851968	1·19458	1·58564	2·06396	2·64046	3·31531
−10	·0317520	·409766	·726887	·994038	1·26847	1·59002	1·97784	2·43694
−11	·0223460	·351656	·630641	·853276	1·05891	1·27929	1·53739	1·84389
−12	·0157488	·303869	·552381	·746687	·914392	1·07568	1·25031	1·45260
−13	·0111102	·263925	·486439	·659789	·805851	·936261	1·06311	1·19929
−14	·00784323	·230142	·429739	·585544	·716682	·830969	·934382	1·03425
−15	—	·201316	·380423	·520659	·639353	·742929	·834951	·918536
−16	—	·176553	·337228	·463376	·570813	·665241	·749524	·825455
−17	—	·155164	·299229	·412568	·509617	·595518	·672811	·742974
−18	—	·136609	·265702	·367403	·454864	·532726	·603281	·667856
−19	—	·120453	·236059	·327212	·405864	·476202	·540292	·599335
−20	—	·106345	·209813	·291427	·362033	·425395	·483373	·537050
−21	—	·093993	·186548	·259560	·322848	·379797	·432074	·480655

TABLE 2 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=2$				
9	135·432	271·364	452·449	678·757	950·304	1267·10	1629·15	2036·45
8	95·6680	191·835	319·901	479·934	671·953	895·965	1151·97	1439·98
7	67·5508	135·600	226·176	339·346	475·129	633·533	814·560	1018·21
6	47·6693	95·8365	159·902	239·935	335·954	447·965	575·973	719·978
5	33·6116	67·7201	113·041	169·641	237·542	316·750	407·267	509·096
4	23·6720	47·8397	79·9048	119·937	167·955	223·966	287·974	359·979
3	16·6448	33·7835	56·4753	84·7911	118·750	158·359	203·621	254·538
2	11·6774	23·8460	39·9096	59·9404	83·9577	111·968	143·976	179·980
1	8·16717	16·8219	28·1978	42·3697	59·3568	79·1660	101·800	127·261
0	5·68831	11·8587	19·9191	29·9474	41·9629	55·9725	71·9787	89·9830
−1	3·94010	8·35431	14·0691	21·1664	29·6658	39·5738	50·8931	63·6248
−2	2·71043	5·88349	9·93802	14·9614	20·9735	27·9806	35·9852	44·9883
−3	1·84991	4·14607	7·02459	10·5795	14·8314	19·7864	25·4464	31·8125
−4	1·25323	2·93032	4·97503	7·48904	10·4945	13·9969	17·9981	22·4988
−5	·845376	2·08652	3·53991	5·31482	7·43628	9·90981	12·7367	15·9174
−6	·571189	1·50762	2·54311	3·79230	5·28557	7·02909	9·02384	11·2697
−7	·388806	1·11507	1·85923	2·73490	3·78062	5·00491	6·40880	7·99187
−8	·267204	·849794	1·39668	2·00998	2·73682	3·59071	4·57388	5·68586
−9	·185173	·667968	1·08631	1·52101	2·02293	2·61254	3·29519	4·07124
−10	·129117	·539190	·875748	1·19497	1·54328	1·94636	2·41452	2·95035
−11	·0904164	·444173	·727559	·975860	1·22529	1·50128	1·81850	2·18307
−12	·0635032	·371392	·617554	·822732	1·01293	1·20792	1·42317	1·66808
−13	·0446919	·313985	·531568	·708691	·864720	1·01250	1·16401	1·32936
−14	·0314975	·267700	·461702	·618220	·753265	·875121	·990905	1·10811
−15	·0222203	·229755	·403444	·543166	·663185	·769720	·866871	·958616
−16	·0156865	·198245	·354022	·479227	·586930	·682434	·768664	·847750
−17	—	·171806	·311603	·423915	·520810	·607028	·685036	·756364
−18	—	·149438	·274894	·375629	·462782	·540661	·611444	·676445
−19	—	·130383	·242935	·333235	·411555	·481787	·545908	·605088
−20	—	·114056	·214986	·295875	·366166	·429386	·487318	·541016
−21	—	·099998	·190460	·262867	·325879	·382684	·434890	·483444
−22	—	—	·16888	·23365	·29008	·34104	—	—

TABLE 2 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s = 0$	1	2	3	4	5	6	7
				$s = 3$				
9	271·279	452·399	678·723	950·281	1267·08	1629·13	2036·43	2488·99
8	191·751	319·851	479·901	671·929	895·947	1151·96	1439·97	1759·97
7	135·515	226·125	339·312	475·105	633·515	814·546	1018·20	1244·48
6	95·7511	159·852	239·901	335·930	447·947	575·959	719·967	879·973
5	67·6338	112·989	169·608	237·518	316·732	407·253	509·084	622·227
4	47·7522	79·8532	119·903	167·931	223·948	287·960	359·968	439·974
3	33·6942	56·4231	84·7568	118·726	158·341	203·607	254·527	311·101
2	23·7544	39·8565	59·9056	83·9332	111·950	143·961	179·969	219·975
1	16·7267	28·1434	42·3344	59·3320	79·1476	101·786	127·249	155·539
0	11·7587	19·8630	29·9113	41·9377	55·9538	71·9643	89·9716	109·977
−1	8·24767	14·0104	21·1292	29·6400	39·5548	50·8785	63·6133	77·7603
−2	5·76755	9·87581	14·9225	20·9469	27·9612	35·9703	44·9766	54·9810
−3	4·01749	6·95743	10·5384	14·8037	19·7663	25·4312	31·8005	38·8753
−4	2·78509	4·90104	7·44481	10·4650	13·9759	17·9823	22·4865	27·4893
−5	1·92054	3·45670	5·26622	7·40458	9·88749	12·7201	15·9046	19·4416
−6	1·31804	2·44798	3·73777	5·25071	7·00496	9·00613	11·2562	13·7558
−7	·902268	1·74977	2·67256	3·74146	4·97827	6·38955	7·97730	9·74211
−8	·618467	1·27185	1·93800	2·69200	3·56067	4·55250	5·66990	6·91343
−9	·426004	·947333	1·43807	1·97102	2·57804	3·27095	4·05339	4·92620
−10	·295267	·726071	1·10097	1·48308	1·90629	2·38656	2·92997	3·53833
−11	·205794	·571915	·872229	1·15614	1·45455	1·78589	2·15945	2·57897
−12	·144044	·460845	·712378	·935156	1·15361	1·38490	1·64039	1·92627
−13	·101130	·377975	·595442	·780134	·950343	1·11897	1·29658	1·49083
−14	·0711527	·314268	·505828	·665056	·806390	·938521	1·06891	1·20439
−15	·0501363	·264110	·434540	·574814	·697352	·808363	·912407	1·01387
−16	·0353643	·223866	·376272	·501127	·609691	·707008	·796131	·879511
−17	·0249630	·191085	·327712	·439342	·536389	·623281	·702447	·775506
−18	—	·164050	·286666	·386643	·473664	·551727	·622949	·688634
−19	—	·141526	·251604	·341182	·419261	·489479	·553735	·613174
−20	—	·122597	·221412	·301658	·371689	·434816	·492755	·546533
−21	—	·106573	·195250	·267107	·329873	·386563	·438725	·487284
−22	—	—	·17247	·23678	·29299	·34384	·39072	—

Table 2 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=4$				
9	452·349	678·690	950·257	1267·06	1629·12	2036·42	2488·98	2986·79
8	319·800	479·867	671·905	895·929	1151·95	1439·96	1759·96	2111·97
7	226·075	339·279	475·081	633·497	814·532	1018·19	1244·47	1493·38
6	159·801	239·868	335·906	447·929	575·945	719·956	879·964	1055·97
5	112·938	169·574	237·494	316·714	407·239	509·073	622·218	746·675
4	79·8011	119·869	167·907	223·930	287·946	359·957	439·965	527·971
3	56·3701	84·7221	118·701	158·323	203·593	254·516	311·092	373·323
2	39·8022	59·8704	83·9084	111·932	143·947	179·958	219·966	263·971
1	28·0874	42·2983	59·3068	79·1290	101·772	127·238	155·530	186·648
0	19·8044	29·8741	41·9119	55·9348	71·9498	89·9601	109·968	131·973
−1	13·9483	21·0904	29·6134	39·5354	50·8637	63·6016	77·7508	93·3125
−2	9·80877	14·8815	20·9191	27·9411	35·9551	44·9646	54·9713	65·9763
−3	6·88348	10·4943	14·7743	19·7453	25·4154	31·7882	38·8654	46·6481
−4	4·81754	7·39625	10·4334	13·9536	17·9658	22·4736	27·4791	32·9829
−5	3·36028	5·21155	7·36973	9·86338	12·7025	15·8910	19·4309	23·3229
−6	2·33470	3·67478	5·21144	6·97831	8·98690	11·2416	13·7444	16·4961
−7	1·61588	2·59869	3·69617	4·94811	6·36815	7·96136	9·72977	11·6743
−8	1·11530	1·85071	2·63882	3·52577	4·52812	5·65202	6·89977	8·27230
−9	·769546	1·33592	1·90812	2·53695	3·24257	4·03285	4·91073	5·87728
−10	·532338	·984617	1·40925	1·85760	2·35304	2·90591	3·52040	4·19806
−11	·369852	·744699	1·07155	1·39728	1·74611	2·13092	2·55783	3·02944
−12	·258118	·578210	·841591	1·08763	1·33801	1·60647	1·90112	2·22616
−13	·180806	·459382	·680732	·876717	1·06465	1·25647	1·46086	1·68349
−14	·127000	·371868	·563494	·727505	·877467	1·02216	1·16876	1·32310
−15	·0893822	·305561	·474403	·616328	·742645	·859612	·972062	1·08441
−16	·0629946	·254116	·404323	·529427	·639485	·739384	·832324	·921002
−17	·0444406	·213416	·347725	·459018	·556538	·644479	·725264	·800614
−18	·0313729	·180695	·301100	·400527	·487580	·566018	·637898	·704536
−19	—	·154032	·262106	·351092	·429024	·499324	·563823	·623646
−20	—	·132059	·229109	·308797	·378619	·441705	·499707	·553626
−21	—	·113774	·200928	·272287	·334840	·391442	·443589	·492185
−22	—	·098431	·17668	·240564	·29658	·34733	·39416	·43790

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 2 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=5$				
9	678·656	950·233	1267·05	1629·11	2036·41	2488·97	2986·78	3529·85
8	479·834	671·881	895·911	1151·93	1439·95	1759·96	2111·96	2495·97
7	339·245	475·057	633·479	814·518	1018·18	1244·46	1493·37	1764·91
6	239·834	335·882	447·911	575·931	719·945	879·955	1055·96	1247·97
5	169·539	237·470	316·695	407·225	509·062	622·209	746·667	882·438
4	119·834	167·882	223·912	287·932	359·945	439·955	527·963	623·969
3	84·6870	118·677	158·304	203·579	254·504	311·083	373·316	441·204
2	59·8346	83·8833	111·913	143·933	179·946	219·956	263·964	311·969
1	42·2615	59·2812	79·1101	101·757	127·227	155·521	186·641	220·587
0	29·8358	41·8856	55·9156	71·9350	89·9484	109·958	131·965	155·971
−1	21·0501	29·5860	39·5155	50·8486	63·5897	77·7412	93·3045	110·280
−2	14·8383	20·8903	27·9204	35·9395	44·9524	54·9615	65·9682	77·9733
−3	10·4470	14·7433	19·7234	25·3990	31·7755	38·8553	46·6398	55·1299
−4	7·34333	10·3995	13·9300	17·9484	22·4603	27·4685	32·9743	38·9787
−5	5·15076	7·33173	9·83749	12·6837	15·8768	19·4197	23·3139	27·5603
−6	3·60323	5·16773	6·94915	8·96614	11·2261	13·7324	16·4865	19·4893
−7	2·51287	3·64470	4·91443	6·34460	7·94403	9·71651	11·6638	13·7868
−8	1·74698	2·57713	3·48596	4·50074	5·63219	6·88482	8·26064	9·76059
−9	1·21166	1·83379	2·48917	3·21005	4·00960	4·89343	5·86396	6·92229
−10	·839896	1·32081	1·80010	2·31391	2·87814	3·49994	4·18248	4·92717
−11	·583190	·969539	1·32908	1·69906	2·09743	2·53325	3·01087	3·53228
−12	·406279	·728894	1·00925	1·28238	1·56629	1·87147	2·20376	2·56618
−13	·284059	·561704	·790325	1·00086	1·20909	1·42534	1·65647	1·90668
−14	·199222	·442620	·636381	·807211	·967921	1·12696	1·29079	1·46421
−15	·140052	·355370	·523932	·668620	·799986	·924288	1·04639	1·17046
−16	·0986241	·289743	·438643	·564593	·676803	·780070	·877710	·972552
−17	·0695355	·239247	·371886	·483173	·581501	·670883	·753759	·831946
−18	·0490685	·199640	·318324	·417396	·504654	·583661	·656429	·724292
−19	·0346466	·168065	·274508	·363022	·440901	·511380	·576234	·636574
−20	·024474	·142540	·238112	·317318	·386987	·450083	·508203	·562327
−21	—	·121661	·207508	·278424	·340794	·397336	·449498	·498161
−22	—	·10440	·181519	·245013	·300849	·351515	·39832	·44207

Table 3. Eigenvalues λ when $\epsilon > 0$ and $s = 1, 2, \ldots, 5$.
Modes travelling westwards, type 1

$g_{\sqrt{2}}(-\eta)$	$\nu = 0$	1	2	3	4	5	6	7
				$s = 1$				
9	−·498896	−45·7581	−135·934	−271·614	−452·599	−678·857	−950·376	−1267·15
8	−·498440	−32·5046	−96·1704	−192·086	−320·051	−480·034	−672·025	−896·018
7	−·497795	−23·1340	−68·0541	−135·851	−226·326	−339·446	−475·201	−633·586
6	−·496885	−16·5092	−48·1740	−96·0878	−160·053	−240·035	−336·025	−448·019
5	−·495602	−11·8266	−34·1182	−67·9719	−113·191	−169·742	−237·614	−316·803
4	−·493793	−8·51795	−24·1814	−48·0922	−80·0557	−120·037	−168·027	−224·020
3	·−491248	−6·18176	−17·1581	−34·0370	−56·6266	−84·8917	−118·822	−158·413
2	−·487679	−4·53426	−12·1963	−24·1011	−40·0614	−60·0412	−84·0295	−112·022
1	−·482695	−3·37497	−8·69382	−17·0790	−28·3504	−42·4708	−59·4288	−79·2199
0	−·475784	−2·56207	−6·22585	−12·1188	−20·0728	−30·0490	−42·0352	−56·0265
−1	−·466312	−1·99492	−4·49272	−8·61871	−14·2243	−21·2687	−29·7384	−39·6281
−2	−·453575	−1·60144	−3·28341	−6·15387	−10·0954	−15·0647	−21·0466	−28·0352
−3	−·436961	−1·32937	−2·44916	−4·42481	−7·18509	−10·6842	−14·9053	−19·8413
−4	−·416252	−1·14037	−1·88381	−3·22061	−5·13985	−7·59565	−10·5693	−14·0524
−5	−·391949	−1·00621	−1·50912	−2·39240	−3·71074	−5·42415	−7·51248	−9·96607
−6	−·365279	−·906095	−1·26504	−1·83395	−2·72225	−3·90541	−5·36372	−7·08646
−7	−·337715	−·824788	−1·10528	−1·46752	−2·04977	−2·85323	−3·86148	−5·06382
−8	−·310424	−·751882	−·995607	−1·23408	−1·60317	−2·13559	−2·82143	−3·65176
−9	−·284115	−·682420	−·911565	−1·08708	−1·31570	−1·65716	−2·11279	−2·67653
−10	−·259159	−·616161	−·836451	−·989248	−1·13708	−1·34752	−1·64097	−2·01455
−11	−·235739	−·554348	−·762503	−·912090	−1·02645	−1·15469	−1·33588	−1·57593
−12	−·213933	−·497627	−·690252	−·837611	−·947797	−1·03854	−1·14621	−1·29387
−13	−·193752	−·446042	−·622071	−·762140	−·874267	−·960552	−1·03371	−1·11995
−14	−·175165	−·399370	−·559047	−·689113	−·797903	−·887832	−·958974	−1·01839
−15	−·158115	−·357287	−·501426	−·620695	−·722838	−·810990	−·886128	−·947664
−16	−·142529	−·319433	−·449099	−·557611	−·651945	−·735160	−·808766	−·873334
−17	−·128322	−·285443	−·401800	−·499996	−·586249	−·663376	−·732892	−·795645
−18	−·115405	−·254964	−·359186	−·447706	−·526042	−·596738	−·661205	−·720298
−19	−·103686	−·227661	−·320885	−·400458	−·471268	−·535592	−·594712	−·649427
−20	−·0930755	−·203224	−·286524	−·357903	−·421690	−·479912	−·533724	−·583854
−21	−·08348	−·181365	−·255740	−·319668	−·376980	−·429482	−·478204	−·523804
−22	—	—	−·22819	—	—	—	—	—

TABLE 3 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	1	2	3	4	5	6	7
				$s = 2$				
9	−·332867	−136·099	−271·697	−452·649	−678·890	−950·400	−1267·17	−1629·20
8	−·332674	−96·3347	−192·168	−320·101	−480·068	−672·048	−896·036	−1152·03
7	−·332402	−68·2175	−135·933	−226·376	−339·479	−475·224	−633·604	−814·615
6	−·332019	−48·3360	−96·1699	−160·102	−240·068	−336·049	−448·037	−576·029
5	−·331479	−34·2782	−68·0534	−113·241	−169·775	−237·637	−316·821	−407·322
4	−·330721	−24·3386	−48·1730	−80·1048	−120·070	−168·050	−224·038	−288·029
3	−·329658	−17·3113	−34·1168	−56·6753	−84·9244	−118·845	−158·431	−203·677
2	−·328173	−12·3438	−24·1794	−40·1096	−60·0737	−84·0529	−112·040	−144·031
1	−·326112	−8·83325	−17·1552	−28·3978	−42·5030	−59·4521	−79·2375	−101·856
0	−·323271	−6·35381	−12·1921	−20·1191	−30·0807	−42·0582	−56·0439	−72·0343
−1	−·319399	−4·60447	−8·68790	−14·2691	−21·2997	−29·7610	−39·6453	−50·9486
−2	−·314204	−3·37262	−6·21733	−10·1381	−15·0947	−21·0687	−28·0521	−36·0408
−3	−·307383	−2·50803	−4·48039	−7·22480	−10·7129	−14·9267	−19·8578	−25·5020
−4	−·298691	−1·90421	−3·26557	−5·17547	−7·62247	−10·5897	−14·0683	−18·0537
−5	−·288025	−1·48494	−2·42344	−3·74084	−5·44837	−7·53157	−9·98125	−12·7923
−6	−·275503	−1·19462	−1·84733	−2·74512	−3·92611	−5·38094	−7·10056	−9·07941
−7	−·261464	−·991860	−1·45891	−2·06363	−2·86934	−3·87616	−5·07643	−6·46439
−8	−·246377	−·845992	−1·19866	−1·60598	−2·14601	−2·83281	−3·66238	−4·62952
−9	−·230712	−·735380	−1·02116	−1·30414	−1·66081	−2·12013	−2·68458	−3·35096
−10	−·214873	−·646360	−·893216	−1·10519	−1·34261	−1·64357	−2·01949	−2·47068
−11	−·199178	−·571474	−·792715	−·968374	−1·13639	−1·33262	−1·57729	−1·87578
−12	−·183863	−·506934	−·707396	−·864528	−·998674	−1·13315	−1·29077	−1·48348
−13	−·169105	−·450613	−·632078	−·776482	−·896340	−1·00191	−1·10854	−1·23160
−14	−·155024	−·401103	−·564814	−·697408	−·808988	−·904302	−·988929	−1·07266
−15	−·141701	−·357366	−·504597	−·625574	−·729157	−·819049	−·897362	−·966759
−16	−·129186	−·318601	−·450676	−·560431	−·655670	−·739739	−·814366	−·880561
−17	−·117499	−·284160	−·402410	−·501547	−·588431	−·666077	−·736097	−·799428
−18	−·106645	−·253510	−·359227	−·448474	−·527277	−·598326	−·663099	−·722489
−19	−·0966089	−·226202	−·320608	−·400751	−·471917	−·536498	−·595825	−·650724
−20	−·0873678	−·201850	−·286087	−·357917	−·421979	−·480397	−·534359	−·584614
−21	−·078889	−·180123	−·255241	−·319524	−·377053	−·429706	−·478542	−·524233
−22	—	−·16073	−·227686	−·28515	−·33672	—	—	—

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Table 3 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	1	2	3	4	5	6	7
				$s = 3$				
9	−·249770	−271·779	−452·699	−678·923	−950·423	−1267·19	−1629·22	−2036·50
8	−·249675	−192·251	−320·151	−480·101	−672·072	−896·054	−1152·04	−1440·03
7	−·249542	−136·015	−226·425	−339·512	−475·248	−633·622	−814·629	−1018·27
6	−·249353	−96·2511	−160·152	−240·101	−336·073	−448·055	−576·042	−720·034
5	−·249087	−68·1338	−113·289	−169·808	−237·661	−316·839	−407·336	−509·151
4	−·248713	−48·2522	−80·1532	−120·103	−168·074	−224·055	−288·043	−360·035
3	−·248189	−34·1942	−56·7231	−84·9568	−118·869	−158·448	−203·691	−254·594
2	−·247456	−24·2543	−40·1565	−60·1057	−84·0760	−112·057	−144·045	−180·036
1	−·246436	−17·2267	−28·4434	−42·5344	−59·4749	−79·2548	−101·869	−127·316
0	−·245026	−12·2586	−20·1630	−30·1113	−42·0806	−56·0610	−72·0477	−90·0383
−1	−·243097	−8·74738	−14·3105	−21·3292	−29·7829	−39·6620	−50·9619	−63·6800
−2	−·240489	−6·26698	−10·1759	−15·1226	−21·0897	−28·0683	−36·0537	−45·0432
−3	−·237024	−4·51636	−7·25754	−10·7385	−14·9465	−19·8734	−25·5145	−31·8672
−4	−·232526	−3·28288	−5·20125	−7·64491	−10·6079	−14·0830	−18·0657	−22·5531
−5	−·226856	−2·41628	−3·75711	−5·46642	−7·54750	−9·99465	−12·8035	−15·9712
−6	−·219951	−1·81013	−2·74877	−3·93819	−5·39371	−7·11215	−9·08949	−11·3229
−7	−·211855	−1·38829	−2·05123	−2·87349	−3·88464	−5·08552	−6·47293	−8·04399
−8	−·202718	−1·09526	−1·57435	−2·14009	−2·83564	−3·66809	−4·63594	−5·73661
−9	−·192756	−·889847	−1·25104	−1·64264	−2·11583	−2·68590	−3·35455	−4·12017
−10	−·182208	−·741926	−1·03022	−1·30990	−1·63062	−2·01534	−2·47064	−2·99694
−11	−·171307	−·630939	−·873958	−1·08687	−1·30911	−1·56654	−1·87130	−2·22699
−12	−·160261	−·544073	−·756471	−·931702	−1·09621	−1·27169	−1·47363	−1·70954
−13	−·149248	−·473736	−·662477	−·815358	−·949342	−1·07790	−1·21459	−1·36970
−14	−·138416	−·415336	−·583951	−·720893	−·838477	−·943862	−1·04479	−1·15008
−15	−·127883	−·365943	−·516707	−·640220	−·746596	−·840434	−·925227	−1·00528
−16	−·117743	−·323585	−·458310	−·569704	−·666448	−·752259	−·829274	−·899200
−17	−·108064	−·286880	−·407165	−·507441	−·595238	−·673790	−·744853	−·809536
−18	−·0988939	−·254824	−·362123	−·452203	−·531610	−·603194	−·668506	−·728500
−19	−·0902642	−·226662	−·322310	−·403079	−·474671	−·539598	−·599242	−·654459
−20	−·0821904	−·201813	−·287026	−·359337	−·423714	−·482370	−·536537	−·586981
−21	−·074676	−·179813	−·255701	−·320360	−·378126	−·430952	−·479930	−·525744
−22	—	−·16029	−·22785	−·28561	−·33736	−·38482	—	—

TABLE 3 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	1	2	3	4	5	6	7
				$s = 4$				
9	−·199872	−452·749	−678·956	−950·447	−1267·21	−1629·23	−2036·51	−2489·05
8	−·199819	−320·200	−480·134	−672·096	−896·072	−1152·06	−1440·05	−1760·04
7	−·199744	−226·475	−339·545	−475·272	−633·640	−814·643	−1018·28	−1244·55
6	−·199638	−160·201	−240·134	−336·096	−448·072	−576·056	−720·045	−880·037
5	−·199489	−113·338	−169·840	−237·684	−316·856	−407·350	−509·162	−622·291
4	−·199280	−80·2011	−120·135	−168·097	−224·073	−288·057	−360·046	−440·037
3	−·198986	−56·7701	−84·9888	−118·892	−158·466	−203·704	−254·605	−311·165
2	−·198575	−40·2022	−60·1371	−84·0988	−112·075	−144·058	−180·047	−220·038
1	−·198001	−28·4874	−42·5650	−59·4973	−79·2718	−101·883	−127·327	−155·603
0	−·197206	−20·2044	−30·1408	−42·1024	−56·0777	−72·0609	−90·0490	−110·040
−1	−·196112	−14·3483	−21·3571	−29·8039	−39·6783	−50·9748	−63·6905	−77·8236
−2	−·194624	−10·2087	−15·1482	−21·1096	−28·0840	−36·0662	−45·0535	−55·0441
−3	−·192628	−7·28324	−10·7609	−14·9648	−19·8881	−25·5265	−31·8770	−38·9381
−4	−·190003	−5·21708	−7·66293	−10·6239	−14·0965	−18·0769	−22·5625	−27·5518
−5	−·186632	−3·75937	−5·47826	−7·56026	−10·0063	−12·8136	−15·9799	−19·5036
−6	−·182428	−2·73293	−3·94154	−5·40202	−7·12122	−9·09803	−11·3305	−13·8171
−7	−·177353	−2·01249	−2·86552	−3·88687	−5·09107	−6·47931	−8·05027	−9·80251
−8	−·171432	−1·50903	−2·11768	−2·82981	−3·66887	−4·63934	−5·74095	−6·97252
−9	−·164750	−1·15844	−1·60305	−2·09978	−2·68042	−3·35395	−4·12185	−4·98351
−10	−·157427	−·913677	−1·25168	−1·60236	−2·00202	−2·46488	−2·99512	−3·59327
−11	−·149608	−·740184	−1·01072	−1·26709	−1·54386	−1·85916	−2·22072	−2·63099
−12	−·141437	−·613660	−·840902	−1·03987	−1·23815	−1·45388	−1·69786	−1·97509
−13	−·133057	−·518163	−·715348	−·880165	−1·03229	−1·18568	−1·35148	−1·53697
−14	−·124596	−·443691	−·617616	−·760578	−·886815	−1·00508	−1·12353	−1·24954
−15	−·116170	−·383964	−·538392	−·665193	−·775644	−·875272	−·968584	−1·06042
−16	−·107879	−·334936	−·472356	−·585732	−·684558	−·772968	−·853568	−·928684
−17	−·0998049	−·293925	−·416263	−·517841	−·606818	−·686655	−·759286	−·826040
−18	−·0920156	−·259093	−·367989	−·458980	−·539121	−·611414	−·677497	−·738398
−19	−·0845631	−·229154	−·326054	−·407492	−·479574	−·544932	−·605000	−·660669
−20	−·0774856	−·203178	−·289379	−·362193	−·426916	−·485855	−·540278	−·590974
−21	−·070809	−·180475	−·257144	−·322189	−·380210	−·433232	−·482377	−·528345
−22	—	−·16052	−·228705	−·286766	−·33871	−·38631	−·43056	−·47209

TABLE 3 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	1	2	3	4	5	6	7
				$s = 5$				
9	−·166588	−678·989	−950·471	−1267·23	−1629·24	−2036·52	−2489·06	−2986·86
8	−·166556	−480·167	−672·119	−896·090	−1152·07	−1440·06	−1760·05	−2112·04
7	−·166510	−339·578	−475·295	−633·657	−814·657	−1018·29	−1244·55	−1493·45
6	−·166445	−240·167	−336·120	−448·090	−576·070	−720·056	−880·046	−1056·04
5	−·166354	−169·873	−237·708	−316·874	−407·364	−509·173	−622·300	−746·743
4	−·166226	−120·167	−168·120	−224·091	−288·071	−360·057	−440·046	−528·039
3	−·166045	−85·0204	−118·915	−158·483	−203·718	−254·615	−311·174	−373·391
2	−·165792	−60·1679	−84·1214	−112·092	−144·072	−180·058	−220·047	−264·039
1	−·165439	−42·5948	−59·5193	−79·2887	−101·896	−127·338	−155·611	−186·716
0	−·164948	−30·1692	−42·1237	−56·0941	−72·0739	−90·0595	−110·049	−132·041
−1	−·164270	−21·3834	−29·8241	−39·6941	−50·9875	−63·7008	−77·8321	−93·3803
−2	−·163343	−15·1716	−21·1283	−28·0990	−36·0784	−45·0635	−55·0524	−66·0440
−3	−·162091	−10·7803	−14·9814	−19·9020	−25·5379	−31·8866	−38·9462	−46·7156
−4	−·160428	−7·67654	−10·6376	−14·1086	−18·0873	−22·5714	−27·5594	−33·0501
−5	−·158264	−5·48384	−7·56983	−10·0161	−12·8226	−15·9879	−19·5106	−23·3897
−6	−·155519	−3·93607	−5·40583	−7·12775	−9·10504	−11·3372	−13·8233	−16·5623
−7	−·152135	−2·84522	−3·88279	−5·09306	−6·48353	−8·05516	−9·80743	−11·7396
−8	−·148093	−2·07841	−2·81521	−3·66467	−4·63972	−5·74334	−6·97575	−8·33640
−9	−·143412	−1·54138	−2·07182	−2·66806	−3·34914	−4·12082	−4·98439	−5·93975
−10	−·138145	−1·16657	−1·55866	−1·97943	−2·45333	−2·98953	−3·59099	−4·25831
−11	−·132373	−·904796	−1·20678	−1·50933	−1·83932	−2·20933	−2·62457	−3·08684
−12	−·126189	−·720047	−·964363	−1·19083	−1·42444	−1·67947	−1·96354	−2·28014
−13	−·119694	−·586822	−·793208	−·972808	−1·14576	−1·32497	−1·51929	−1·73399
−14	−·112989	−·488014	−·667292	−·817811	−·954887	−1·08793	−1·22463	−1·37100
−15	−·106170	−·412559	−·570488	−·701267	−·817004	−·923960	−1·02719	−1·13148
−16	−·0993285	−·353335	−·493279	−·608929	−·710401	−·802229	−·887448	−·968812
−17	−·0925453	−·305697	−·429967	−·532970	−·623382	−·704881	−·779591	−·849068
−18	−·0858917	−·266555	−·376972	−·468927	−·549919	−·623094	−·690178	−·752286
−19	−·0794276	−·233816	−·331927	−·414055	−·486684	−·552553	−·613151	−·669407
−20	−·0732023	−·206026	−·293195	−·366523	−·431620	−·490880	−·545610	−·596620
−21	−·067254	−·182157	−·259597	−·325034	−·383324	−·436562	−·485898	−·532050
−22	—	−·161462	−·230257	−·288623	−·340763	−·388517	−·43290	−·47454

Table 4. Eigenvalues λ when $\epsilon > 0$ and $s = 1, 2, \ldots, 5$.
Modes travelling westwards, class 2

$\log_{\sqrt{2}}(-\eta)$	$n-s=0$	1	2	3	4	5	6	7
				$s = 1$				
24	−·499994	−·166644	−·0833223	−·0499936	−·0333291	−·0238065	−·0178549	−·0138872
23	−·499991	−·166635	−·0833178	−·0499909	−·0333274	−·0238053	−·0178540	−·0138865
22	−·499988	−·166622	−·0833114	−·0499871	−·0333249	−·0238036	−·0178527	−·0138854
21	−·499983	−·166603	−·0833022	−·0499818	−·0333214	−·0238011	−·0178509	−·0138840
20	−·499976	−·166577	−·0832894	−·0499743	−·0333164	−·0237976	−·0178482	−·0138820
19	−·499966	−·166540	−·0832712	−·0499636	−·0333094	−·0237926	−·0178446	−·0138791
18	−·499951	−·166487	−·0832454	−·0499485	−·0332995	−·0237856	−·0178393	−·0138751
17	−·499931	−·166413	−·0832090	−·0499272	−·0332855	−·0237757	−·0178320	−·0138694
16	−·499902	−·166308	−·0831575	−·0498970	−·0332657	−·0237617	−·0178215	−·0138613
15	−·499862	−·166159	−·0830846	−·0498544	−·0332377	−·0237419	−·0178068	−·0138499
14	−·499805	−·165949	−·0829817	−·0497941	−·0331981	−·0237139	−·0177859	−·0138337
13	−·499724	−·165653	−·0828360	−·0497087	−·0331421	−·0236743	−·0177563	−·0138108
12	−·499610	−·165234	−·0826301	−·0495881	−·0330629	−·0236182	−·0177146	−·0137785
11	−·499448	−·164643	−·0823390	−·0494174	−·0329508	−·0235389	−·0176554	−·0137327
10	−·499219	−·163809	−·0819274	−·0491760	−·0327922	−·0234266	−·0175718	−·0136679
9	−·498896	−·162634	−·0813457	−·0488345	−·0325677	−·0232678	−·0174533	−·0135762
8	−·498440	−·160981	−·0805238	−·0483514	−·0322500	−·0230428	−·0172856	−·0134462
7	−·497795	−·158661	−·0793631	−·0476677	−·0318000	−·0227240	−·0170478	−·0132620
6	−·496885	−·155417	−·0777252	−·0467002	−·0311623	−·0222720	−·0167105	−·0130006
5	−·495602	−·150903	−·0754173	−·0453303	−·0302577	−·0216301	−·0162312	−·0126290
4	−·493793	−·144671	−·0721738	−·0433902	−·0289723	−·0207163	−·0155483	−·0120991
3	−·491248	−·136171	−·0676391	−·0406426	−·0271409	−·0194106	−·0145707	−·0113399
2	−·487679	−·124802	−·0613691	−·0367600	−·0245227	−·0175322	−·0131595	−·0102414
1	−·482695	−·110082	−·0529202	−·0313436	−·0207823	−·0148070	−·0110916	−·00862147
0	−·475784	−·0920869	−·0422658	−·0242477	−·0157030	−·0109875	−·00811378	−·00623462
−1	−·466312	−·0722274	−·0308711	−·0168469	−·0105216	−·00716564	−·00518312	−·00391859
−2	−·453575	−·0535315	−·0214821	−·0113731	−·00699373	−·00472217	−·00339804	−·00256049
−3	−·436961	−·0385340	−·0148900	−·00777310	−·00474932	−·00319577	−·00229500	−·00172709
−4	−·416252	−·0274644	−·0103711	−·00537142	−·00327039	−·00219656	−·00157573	—
−5	−·391949	−·0195024	−·00725501	−·00373915	−·00227172	−·0015241	—	—
−6	−·365279	−·0138245	−·00509130	−·00261564	−·0015870	—	—	—
−7	−·337715	−·00979045	−·00358093	−·0018358	—	—	—	—
−8	−·310424	−·00692978	−·0025226	—	—	—	—	—
−9	−·284115	−·00490332	—	—	—	—	—	—
−10	−·259159	−·0034687	—	—	—	—	—	—
−11	−·235739	—	—	—	—	—	—	—
−12	−·213933	—	—	—	—	—	—	—
−13	−·193752	—	—	—	—	—	—	—
−14	−·175165	—	—	—	—	—	—	—
−15	−·158115	—	—	—	—	—	—	—
−16	−·142529	—	—	—	—	—	—	—
−17	−·128322	—	—	—	—	—	—	—
−18	−·115405	—	—	—	—	—	—	—
−19	−·103686	—	—	—	—	—	—	—
−20	−·093075							

TABLE 4 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$n-s=0$	1	2	3	4	5	6	7
				$s=2$				
24	−·333331	−·166659	−·0999947	−·0666629	−·0476163	−·0357122	−·0277761	−·0222209
23	−·333330	−·166656	−·0999925	−·0666614	−·0476152	−·0357113	−·0277755	−·0222204
22	−·333328	−·166652	−·0999894	−·0666592	−·0476135	−·0357101	−·0277745	−·0222196
21	−·333326	−·166646	−·0999850	−·0666561	−·0476113	−·0357084	−·0277731	−·0222185
20	−·333323	−·166637	−·0999788	−·0666517	−·0476080	−·0357059	−·0277712	−·0222169
19	−·333319	−·166625	−·0999701	−·0666455	−·0476035	−·0357024	−·0277685	−·0222147
18	−·333313	−·166608	−·0999577	−·0666367	−·0475970	−·0356975	−·0277646	−·0222116
17	−·333304	−·166584	−·0999402	−·0666242	−·0475879	−·0356905	−·0277591	−·0222072
16	−·333292	−·166549	−·0999154	−·0666067	−·0475750	−·0356807	−·0277514	−·0222010
15	−·333275	−·166501	−·0998803	−·0665818	−·0475567	−·0356668	−·0277405	−·0221922
14	−·333251	−·166432	−·0998307	−·0665467	−·0475309	−·0356471	−·0277250	−·0221798
13	−·333217	−·166334	−·0997606	−·0664969	−·0474943	−·0356193	−·0277032	−·0221622
12	−·333168	−·166197	−·0996615	−·0664266	−·0474426	−·0355799	−·0276723	−·0221373
11	−·333100	−·166002	−·0995214	−·0663272	−·0473696	−·0355242	−·0276285	−·0221021
10	−·333003	−·165728	−·0993232	−·0661866	−·0472662	−·0354454	−·0275667	−·0220523
9	−·332867	−·165340	−·0990431	−·0659878	−·0471200	−·0353340	−·0274791	−·0219818
8	−·332674	−·164792	−·0986471	−·0657066	−·0469131	−·0351764	−·0273553	−·0218821
7	−·332402	−·164019	−·0980876	−·0653090	−·0466205	−·0349533	−·0271800	−·0217409
6	−·332019	−·162929	−·0972972	−·0647467	−·0462065	−·0346375	−·0269319	−·0215410
5	−·331479	−·161394	−·0961816	−·0639517	−·0456206	−·0341903	−·0265803	−·0212578
4	−·330721	−·159239	−·0946082	−·0628280	−·0447912	−·0335568	−·0260820	−·0208561
3	−·329658	−·156221	−·0923934	−·0612403	−·0436169	−·0326586	−·0253748	−·0202857
2	−·328173	−·152017	−·0892851	−·0589997	−·0419538	−·0313837	−·0243695	−·0194740
1	−·326112	−·146206	−·0849478	−·0558464	−·0395996	−·0295719	−·0229371	−·0183153
0	−·323271	−·138275	−·0789633	−·0514393	−·0362764	−·0269963	−·0208903	−·0166536
−1	−·319399	−·127687	−·0708959	−·0453963	−·0316464	−·0233594	−·0179692	−·0142617
−2	−·314204	−·114074	−·0605501	−·0375521	−·0255385	−·0184787	−·0139821	−·0109441
−3	−·307383	−·0976605	−·0485727	−·0287035	−·0187925	−·0131868	−·00972992	−·0074580
−4	−·298691	−·0797349	−·0368014	−·0207442	−·0131720	−·00905670	−·00659017	−·0050020
−5	−·288025	−·0623878	−·0269451	−·0146940	−·00916557	−·00623696	−·00450909	−·0034079
−6	−·275503	−·0473012	−·0194102	−·0103595	−·00639410	−·00432584	−·00311649	—
−7	−·261464	−·0350983	−·0138725	−·00729959	−·00447591	−·0030175	—	—
−8	−·246377	−·0256643	−·00987336	−·00514608	−·0031420	—	—	—
−9	−·230712	−·0185777	−·00701049	−·0036303	—	—	—	—
−10	−·214873	−·0133542	−·00497064	—	—	—	—	—
−11	−·199178	−·00955262	—	—	—	—	—	—
−12	−·183863	−·0068100	—	—	—	—	—	—
−13	−·169105	—	—	—	—	—	—	—
−14	−·155024	—	—	—	—	—	—	—
−15	−·141701	—	—	—	—	—	—	—
−16	−·129186	—	—	—	—	—	—	—
−17	−·117499	—	—	—	—	—	—	—
−18	−·106645	—	—	—	—	—	—	—
−19	−·0966089	—	—	—	—	—	—	—
−20	−·0873678	—	—	—	—	—	—	—
−21	−·078889	—	—	—	—	—	—	—

TABLE 4 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$n-s=0$	1	2	3	4	5	6	7
				$s=3$				
24	−·249999	−·149997	−·0999970	−·0714262	−·0535695	−·0416651	−·0333321	−·0272717
23	−·249998	−·149995	−·0999958	−·0714252	−·0535688	−·0416645	−·0333316	−·0272713
22	−·249998	−·149993	−·0999941	−·0714238	−·0535677	−·0416636	−·0333308	−·0272707
21	−·249996	−·149990	−·0999916	−·0714219	−·0535661	−·0416624	−·0333298	−·0272698
20	−·249995	−·149987	−·0999882	−·0714191	−·0535639	−·0416606	−·0333283	−·0272686
19	−·249993	−·149981	−·0999833	−·0714152	−·0535608	−·0416581	−·0333263	−·0272669
18	−·249990	−·149973	−·0999763	−·0714096	−·0535563	−·0416545	−·0333234	−·0272644
17	−·249986	−·149962	−·0999665	−·0714018	−·0535501	−·0416494	−·0333192	−·0272610
16	−·249980	−·149946	−·0999526	−·0713907	−·0535412	−·0416423	−·0333134	−·0272561
15	−·249971	−·149924	−·0999330	−·0713750	−·0535287	−·0416322	−·0333051	−·0272492
14	−·249959	−·149892	−·0999053	−·0713527	−·0535110	−·0416179	−·0332934	−·0272395
13	−·249943	−·149847	−·0998660	−·0713213	−·0534860	−·0415978	−·0332769	−·0272258
12	−·249919	−·149784	−·0998106	−·0712769	−·0534506	−·0415692	−·0332535	−·0272063
11	−·249885	−·149694	−·0997321	−·0712141	−·0534006	−·0415288	−·0332204	−·0271788
10	−·249838	−·149568	−·0996212	−·0711253	−·0533298	−·0414717	−·0331736	−·0271398
9	−·249770	−·149389	−·0994644	−·0709997	−·0532297	−·0413909	−·0331074	−·0270848
8	−·249675	−·149136	−·0992428	−·0708221	−·0530881	−·0412767	−·0330138	−·0270069
7	−·249542	−·148779	−·0989296	−·0705711	−·0528879	−·0411151	−·0328813	−·0268967
6	−·249353	−·148276	−·0984872	−·0702162	−·0526048	−·0408865	−·0326939	−·0267407
5	−·249087	−·147566	−·0978625	−·0697146	−·0522044	−·0405631	−·0324287	−·0265199
4	−·248713	−·146567	−·0969813	−·0690061	−·0516382	−·0401055	−·0320532	−·0262072
3	−·248189	−·145162	−·0957395	−·0680057	−·0508378	−·0394579	−·0315215	−·0257643
2	−·247456	−·143195	−·0939933	−·0665947	−·0497065	−·0385414	−·0307682	−·0251362
1	−·246436	−·140450	−·0915456	−·0646085	−·0481090	−·0372442	−·0297003	−·0242446
0	−·245026	−·136647	−·0881342	−·0618235	−·0458581	−·0354096	−·0281857	−·0229776
−1	−·243097	−·131432	−·0834282	−·0579493	−·0427033	−·0328225	−·0260395	−·0211752
−2	−·240489	−·124402	−·0770596	−·0526554	−·0383469	−·0292149	−·0230209	−·0186213
−3	−·237024	−·115179	−·0687508	−·0457132	−·0325815	−·0243888	−·0189366	−·0151267
−4	−·232526	−·103595	−·0586207	−·0373953	−·0257584	−·0187319	−·0141865	−·0110888
−5	−·226856	−·0899609	−·0475430	−·0288494	−·0191388	−·0135266	−·0100229	−·00770299
−6	−·219951	−·0752223	−·0369155	−·0213922	−·0137818	−·00955574	−·00698968	−·00532364
−7	−·211855	−·0606799	−·0277858	−·0155317	−·00980785	−·00671992	−·00487854	−·0036971
−8	−·202718	−·0474560	−·0204895	−·0111545	−·00694985	−·00472587	—	—
−9	−·192756	−·0361906	−·0149076	−·00796349	−·00491687	—	—	—
−10	−·182208	−·0270598	−·0107500	−·00566584	—	—	—	—
−11	−·171307	−·0199303	−·00770537	—	—	—	—	—
−12	−·160261	−·0145148	—	—	—	—	—	—
−13	−·149248	−·0104836	—	—	—	—	—	—
−14	−·138416	—	—	—	—	—	—	—
−15	−·127883	—	—	—	—	—	—	—
−16	−·117743	—	—	—	—	—	—	—
−17	−·108064	—	—	—	—	—	—	—
−18	−·0988939	—	—	—	—	—	—	—
−19	−·0902642	—	—	—	—	—	—	—
−20	−·0821904	—	—	—	—	—	—	—
−21	−·074676	—	—	—	—	—	—	—

TABLE 4 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$n-s=0$	1	2	3	4	5	6	7
				$s=4$				
24	−·199999	−·133332	−·0952363	−·0714270	−·0555542	−·0444433	−·0363627	−·0303022
23	−·199999	−·133331	−·0952355	−·0714263	−·0555537	−·0444428	−·0363623	−·0303019
22	−·199999	−·133330	−·0952344	−·0714254	−·0555529	−·0444422	−·0363617	−·0303014
21	−·199998	−·133328	−·0952329	−·0714241	−·0555518	−·0444412	−·0363609	−·0303007
20	−·199997	−·133326	−·0952308	−·0714222	−·0555502	−·0444399	−·0363598	−·0302997
19	−·199996	−·133323	−·0952278	−·0714196	−·0555479	−·0444380	−·0363582	−·0302984
18	−·199994	−·133319	−·0952235	−·0714158	−·0555448	−·0444354	−·0363559	−·0302964
17	−·199992	−·133312	−·0952174	−·0714106	−·0555403	−·0444316	−·0363527	−·0302937
16	−·199989	−·133304	−·0952088	−·0714031	−·0555340	−·0444263	−·0363482	−·0302898
15	−·199984	−·133291	−·0951967	−·0713926	−·0555251	−·0444187	−·0363418	−·0302844
14	−·199977	−·133274	−·0951796	−·0713776	−·0555125	−·0444081	−·0363328	−·0302767
13	−·199968	−·133249	−·0951553	−·0713565	−·0554946	−·0443930	−·0363200	−·0302657
12	−·199955	−·133215	−·0951211	−·0713267	−·0554694	−·0443717	−·0363019	−·0302503
11	−·199936	−·133166	−·0950726	−·0712845	−·0554337	−·0443416	−·0362764	−·0302284
10	−·199909	−·133096	−·0950041	−·0712249	−·0553832	−·0442990	−·0362402	−·0301975
9	−·199872	−·132998	−·0949072	−·0711405	−·0553118	−·0442387	−·0361891	−·0301538
8	−·199819	−·132859	−·0947703	−·0710213	−·0552108	−·0441535	−·0361168	−·0300919
7	−·199744	−·132663	−·0945768	−·0708527	−·0550681	−·0440330	−·0360145	−·0300044
6	−·199638	−·132387	−·0943035	−·0706144	−·0548662	−·0438626	−·0358698	−·0298806
5	−·199489	−·131997	−·0939176	−·0702778	−·0545809	−·0436215	−·0356652	−·0297055
4	−·199280	−·131447	−·0933730	−·0698023	−·0541777	−·0432807	−·0353757	−·0294578
3	−·198986	−·130673	−·0926056	−·0691312	−·0536079	−·0427988	−·0349662	−·0291071
2	−·198575	−·129588	−·0915255	−·0681850	−·0528034	−·0421176	−·0343867	−·0286106
1	−·198001	−·128070	−·0900096	−·0668530	−·0516685	−·0411550	−·0335670	−·0279074
0	−·197206	−·125956	−·0878904	−·0649837	−·0500705	−·0397962	−·0324075	−·0269114
−1	−·196112	−·123035	−·0849479	−·0623742	−·0478294	−·0378831	−·0307699	−·0255010
−2	−·194624	−·119043	−·0809088	−·0587688	−·0447124	−·0352066	−·0284668	−·0235088
−3	−·192628	−·113679	−·0754747	−·0538890	−·0404611	−·0315264	−·0252755	−·0207281
−4	−·190003	−·106654	−·0684170	−·0475548	−·0349245	−·0267075	−·0210687	−·0170353
−5	−·186632	−·0977977	−·0597779	−·0399634	−·0284019	−·0211146	−·0162512	−·0128581
−6	−·182428	−·0872119	−·0500926	−·0319204	−·0218461	−·0157641	−·0118498	−·009200
−7	−·177353	−·0753733	−·0403159	−·0244710	−·0161966	−·0114186	−·00844319	−·006478
−8	−·171432	−·0630657	−·0313666	−·0182448	−·0117688	−·00816389	−·00597279	−·004549
−9	−·164750	−·0511477	−·0237737	−·0133623	−·00845952	−·00580406	—	—
−10	−·157427	−·0403229	−·0176718	−·00967484	−·0060439	—	—	—
−11	−·149608	−·0310186	−·0129511	−·0069526	—	—	—	—
−12	−·141437	−·0233806	−·00939459	—	—	—	—	—
−13	−·133057	−·0173392	—	—	—	—	—	—
−14	−·124596	−·0126977	—	—	—	—	—	—
−15	−·116170	—	—	—	—	—	—	—
−16	−·107879	—	—	—	—	—	—	—
−17	−·0998049	—	—	—	—	—	—	—
−18	−·0920156	—	—	—	—	—	—	—
−19	−·0845631	—	—	—	—	—	—	—
−20	−·0774856	—	—	—	—	—	—	—
−21	−·070809	—	—	—	—	—	—	—

TABLE 4 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$n-s=0$	1	2	3	4	5	6	7
				$s = 5$				
24	−·166666	−·119047	−·0892845	−·0694433	−·0555546	−·0454537	−·0378780	−·0320506
23	−·166666	−·119046	−·0892840	−·0694429	−·0555542	−·0454533	−·0378777	−·0320503
22	−·166666	−·119045	−·0892833	−·0694422	−·0555536	−·0454528	−·0378773	−·0320500
21	−·166665	−·119044	−·0892823	−·0694413	−·0555528	−·0454521	−·0378766	−·0320494
20	−·166665	−·119043	−·0892809	−·0694400	−·0555516	−·0454511	−·0378758	−·0320486
19	−·166664	−·119041	−·0892789	−·0694381	−·0555499	−·0454496	−·0378745	−·0320475
18	−·166663	−·119039	−·0892760	−·0694355	−·0555476	−·0454476	−·0378727	−·0320460
17	−·166662	−·119035	−·0892720	−·0694318	−·0555443	−·0454447	−·0378702	−·0320438
16	−·166660	−·119030	−·0892663	−·0694265	−·0555397	−·0454406	−·0378666	−·0320407
15	−·166657	−·119022	−·0892583	−·0694191	−·0555331	−·0454349	−·0378616	−·0320362
14	−·166653	−·119011	−·0892470	−·0694086	−·0555238	−·0454267	−·0378545	−·0320300
13	−·166647	−·118996	−·0892309	−·0693937	−·0555106	−·0454152	−·0378444	−·0320212
12	−·166639	−·118975	−·0892082	−·0693727	−·0554920	−·0453989	−·0378302	−·0320088
11	−·166627	−·118945	−·0891761	−·0693430	−·0554656	−·0453759	−·0378101	−·0319911
10	−·166611	−·118903	−·0891308	−·0693010	−·0554284	−·0453433	−·0377816	−·0319662
9	−·166588	−·118843	−·0890666	−·0692416	−·0553757	−·0452972	−·0377414	−·0319310
8	−·166556	−·118758	−·0889759	−·0691577	−·0553013	−·0452320	−·0376845	−·0318811
7	−·166510	−·118638	−·0888478	−·0690390	−·0551960	−·0451399	−·0376040	−·0318106
6	−·166445	−·118469	−·0886668	−·0688713	−·0550472	−·0450096	−·0374901	−·0317109
5	−·166354	−·118230	−·0884113	−·0686344	−·0548369	−·0448254	−·0373292	−·0315699
4	−·166226	−·117894	−·0880506	−·0682998	−·0545397	−·0445650	−·0371015	−·0313705
3	−·166045	−·117420	−·0875422	−·0678276	−·0541199	−·0441969	−·0367797	−·0310883
2	−·165792	−·116755	−·0868265	−·0671619	−·0535274	−·0436770	−·0363246	−·0306893
1	−·165439	−·115823	−·0858213	−·0662248	−·0526920	−·0429429	−·0356815	−·0301248
0	−·164948	−·114522	−·0844140	−·0649090	−·0515162	−·0419077	−·0347732	−·0293265
−1	−·164270	−·112718	−·0824543	−·0630696	−·0498667	−·0404514	−·0334922	−·0281984
−2	−·163343	−·110237	−·0797481	−·0605170	−·0475669	−·0384123	−·0316922	−·0266084
−3	−·162091	−·106869	−·0760625	−·0570226	−·0444003	−·0355883	−·0291858	−·0243833
−4	−·160428	−·102381	−·0711569	−·0523585	−·0401528	−·0317775	−·0257814	−·0213410
−5	−·158264	−·0965561	−·0648671	−·0464168	−·0347582	−·0269419	−·0214587	−·0174713
−6	−·155519	−·0892726	−·0572535	−·0394193	−·0285610	−·0215148	−·0167141	−·0133138
−7	−·152135	−·0805912	−·0487293	−·0320230	−·0223551	−·0163479	−·0124032	−·00969458
−8	−·148093	−·0708133	−·0399943	−·0250369	−·0168845	−·0120465	−·00897840	−·00692660
−9	−·143412	−·0604500	−·0317655	−·0190117	−·0124620	−·00872869	−·00642549	—
−10	−·138145	−·0501132	−·0245419	−·0141363	−·00906539	−·0062659	—	—
−11	−·132373	−·0403838	−·0185436	−·0103561	−·0065337	—	—	—
−12	−·126189	−·0317046	−·0137702	−·0075080	—	—	—	—
−13	−·119694	−·0243257	−·010091	—	—	—	—	—
−14	−·112989	−·0183071	—	—	—	—	—	—
−15	−·106170	−·013564	—	—	—	—	—	—
−16	−·0993285	—	—	—	—	—	—	—
−17	−·0925453	—	—	—	—	—	—	—
−18	−·0858917	—	—	—	—	—	—	—
−19	−·0794276	—	—	—	—	—	—	—
−20	−·0732023	—	—	—	—	—	—	—
−21	−·067254	—	—	—	—	—	—	—

Table 5. Eigenvalues λ when $\epsilon > 0$ and $s = 0, 1, 2, \ldots, 5$, interpolated at given values of ϵ

	$\epsilon = 10^4$	10^3	10^2	10	1	10^{-1}	10^{-2}	
				$s = 0$				
$\nu = 6$	±·35761	±·62393	±1·0462	±2·4695	±7·51623	±23·6747	±74·8364	$n-s = 7$
5	·32941	·57664	·97274	2·1674	6·51856	20·5059	64·8112	6
4	·29836	·52400	·89316	1·8697	5·52160	17·3346	54·7767	5
3	·26348	·46424	·80067	1·5782	4·52567	14·1591	44·7267	4
2	·22299	·39416	·68776	1·2925	3·53099	10·9757	34·6477	3
1	·17299	·30682	·54165	·99580	2·53470	7·77356	24·5036	2
0	·10013	·17856	·32075	·60365	1·48174	4·49439	14·1492	1
$\nu'' = 0$	·00000	·00000	·00000	·00000	0·00000	0·00000	0·0000	0
				$s = 1$				
$\nu = 6$	·35821	·62702	1·09879	2·76592	8·50701	26·83496	84·84876	$n-s = 7$
5	·33008	·57992	1·01556	2·45780	7·50663	23·66562	74·82746	6
4	·29915	·52759	·92786	2·15098	6·50560	20·49367	64·79920	5
3	·26445	·46836	·83052	1·84500	5·50316	17·31741	54·75990	4
2	·22427	·39927	·71662	1·53721	4·49732	14·13320	44·70146	3
1	·17498	·31419	·57504	1·21665	3·48187	10·93212	34·60551	2
0	·10529	·19550	·37963	·84590	2·43159	7·68509	24·41884	1
$\nu'' = 0$	·010025	·031877	·10263	·34457	1·23068	4·24517	13·89959	0
$\nu = 6$	−·35732	−·62292	−1·00837	−2·17787	−6·52971	−20·51751	−64·82301	$n-s = 5$
5	−·32905	−·57549	−·95051	−1·88459	−5·53707	−17·35080	−54·79322	4
4	−·29791	−·52264	−·88030	−1·60193	−4·54865	−14·18334	−44·75147	3
3	−·26290	−·46251	−·79130	−1·34003	−3·56900	−11·01583	−34·68888	2
2	−·22216	−·39171	−·67845	−1·11186	−2·61292	−7·85329	−24·58566	1
1	−·17154	−·30258	−·52836	−·88188	−1·74147	−4·74640	−14·39971	0
0	−·095215	−·16309	−·27096	−·41399	−·48797	−·49875	−·49988	$n'-s = 0$
$\nu' = 1$	−·0033317	−·010523	−·033085	−·094951	−·15297	−·16515	−·16651	1
2	−·0020071	−·0063967	−·020706	−·058026	−·079750	−·082960	−·083296	2
3	−·0014385	−·0046186	−·015349	−·039534	−·048715	−·049869	−·049987	3
4	−·0011221	−·0036284	−·012368	−·028377	−·032766	−·033276	−·033328	4
5	−·00092084	−·00299800	−·01045	−·021192	−·023521	−·023780	−·023807	5
				$s = 2$				
$\nu = 6$	·35911	·63213	1·16127	3·06661	9·50061	29·99678	94·85972	$n-s = 7$
5	·33107	·58527	1·07154	2·75461	8·49881	26·82762	84·84169	6
4	·30028	·53335	·97793	2·44243	7·49586	23·65612	74·81838	5
3	·26579	·47480	·87656	2·12905	6·49086	20·48090	64·78703	4
2	·22598	·40691	·76174	1·81173	5·48181	17·29931	54·74279	3
1	·17747	·32438	·62445	1·48294	4·46384	14·10562	44·67565	2
0	·11071	·21388	·44708	1·12243	3·42299	10·88512	34·56217	1
$\nu'' = 0$	·020050	·063753	·20520	·68283	2·31222	7·58910	24·33137	0
$\nu = 6$	−·35732	−·62399	−1·05136	−2·48130	−7·53188	−23·69186	−74·85405	$n-s = 5$
5	−·32902	−·57650	−·97815	−2·18220	−6·53902	−20·52858	−64·83465	4
4	−·29782	−·52355	−·89834	−1·88880	−5·54950	−17·36608	−54·80943	3
3	−·26272	−·46323	−·80506	−1·60359	−4·56600	−14·20584	−44·77567	2
2	−·22178	−·39206	−·69022	−1·32654	−3·59452	−11·05228	−34·72888	1
1	−·17066	−·30176	−·53809	−1·03662	−2·64888	−7·92278	−2·46647	0
0	−·090556	−·14907	−·22997	−·30561	−·32990	−·33298	−·33330	$n'-s = 0$
$\nu' = 1$	−·0066038	−·020441	−·059794	−·13042	−·16184	−·16617	−·16662	1
2	−·0039906	−·012542	−·038242	−·082513	−·097883	−·099784	−·099978	2
3	−·0028644	−·0090976	−·028507	−·057808	−·065658	−·066564	−·066656	3
4	−·0022365	−·0071660	−·022886	−·042775	−·047087	−·047565	−·047614	4
5	−·0018361	−·0059307	−·019149	−·032874	−·035410	−·035684	−·035711	5

TABLE 5 (*cont.*)

	$\epsilon = 10^4$	10^3	10^2	10	1	10^{-1}	10^{-2}	
				$s = 3$				
$\nu = 6$	·36030	·63920	1·23100	3·37065	10·49612	33·15947	104·8694	$n-s = 7$
5	·33238	·59263	1·13684	3·05616	9·49358	29·99076	94·85401	6
4	·30175	·54120	1·03912	2·74082	8·48979	26·82003	84·83455	5
3	·26751	·48343	·93486	2·42354	7·48394	23·64626	74·80915	4
2	·22811	·41691	·81956	2·10200	6·47437	20·46756	64·77467	3
1	·18044	·33705	·68610	1·77090	5·45764	17·28029	54·72538	2
0	·11637	·23366	·52197	1·41797	4·42553	14·07639	44·64932	1
$\nu'' = 0$	·030075	·095629	·30769	1·01549	3·35560	10·83485	34·51774	0
$\nu = 6$	−·35763	−·62714	−1·10756	−2·78512	−8·53172	−26·86173	−84·87618	$n-s = 5$
5	−·32930	−·57965	−1·02479	−2·48128	−7·53792	−23·69987	−74·86272	4
4	−·29809	−·52671	−·93702	−2·18028	−6·54649	−20·53906	−64·84610	3
3	−·26293	−·46643	−·83865	−1·88228	−5·55885	−17·38041	−54·82538	2
2	−·22187	−·39523	−·72174	−1·58501	−4·57756	−14·22660	−44·79934	1
1	−·17037	−·30455	−·57068	−1·27502	−3·60686	−11·08498	−34·76775	0
0	−·086147	−·13645	−·19684	−·23883	−·24872	−·24987	−·24999	$n'-s = 0$
$\nu' = 1$	−·0097593	−·029237	−·076911	−·13253	−·14796	−·14979	−·14998	1
2	−·0059277	−·018215	−·050922	−·089365	−·098804	−·099879	−·099988	2
3	−·0042656	−·013311	−·038410	−·065141	−·070742	−·071359	−·071422	3
4	−·0033355	−·010529	−·030902	−·049728	−·053160	−·053530	−·053567	4
5	−·0027410	−·0087377	−·025791	−·039216	−·041408	−·041641	−·041664	5
				$s = 4$				
$\nu = 6$	·36177	·64816	1·30627	3·67725	11·49292	36·32266	114·8780	$n-s = 7$
5	·33399	·60191	1·20904	3·36116	10·49005	33·15446	104·8647	6
4	·30355	·55101	1·10852	3·04395	9·48597	29·98456	94·84826	5
3	·26959	·49413	1·00236	2·72462	8·47999	26·81219	84·82730	4
2	·23064	·42907	·88702	2·40131	7·47086	23·63602	74·79977	3
1	·18385	·35194	·75693	2·07021	6·45615	20·45366	64·76213	2
0	·12229	·25476	·60291	1·72341	5·43074	17·26036	54·70768	1
$\nu'' = 0$	·040100	·12750	·41004	1·34414	4·38266	14·04554	44·62245	0
$\nu = 6$	−·35823	−·63235	−1·17266	−3·09047	−9·53062	−30·02903	−94·89271	$n-s = 5$
5	−·32992	−·58492	−1·08358	−2·78323	−8·53586	−26·86778	−84·88284	4
4	−·29871	−·53212	−·99010	−2·47729	−7·54275	−23·70750	−74·87124	3
3	−·26354	−·47206	−·88780	−2·17217	−6·55210	−20·54897	−64·85738	2
2	−·22243	−·40121	−·76977	−1·86553	−5·56507	−17·39380	−54·84099	1
1	−·17069	−·31104	−·62248	−1·54887	−4·58323	−14·24560	−44·82245	0
0	−·081982	−·12515	−·17028	−·19475	−·19942	−·19994	−·19999	0
$\nu' = 1$	−·012746	−·036556	−·085152	−·12424	−·13233	−·13323	−·13332	$n'-s = 1$
2	−·0077971	−·023246	−·058754	−·088704	−·094531	−·095167	−·095231	2
3	−·0056305	−·017156	−·045060	−·067086	−·070966	−·071382	−·071424	3
4	−·0044116	−·013649	−·036505	−·052658	−·055251	−·055525	−·055552	4
5	−·0036303	−·011367	−·030552	−·042468	−·044239	−·044424	·044442	5
				$s = 5$				
$\nu = 6$	·36353	·65891	1·38587	3·98580	12·49064	39·48610	124·8856	$n-s = 7$
5	·33590	·61301	1·28648	3·66869	11·48765	36·31841	114·8741	6
4	·30568	·56269	1·18407	3·35038	10·48356	33·14930	104·8600	5
3	·27202	·50674	1·07676	3·03001	9·47779	29·97818	94·84243	4
2	·23357	·44320	·96172	2·70611	8·46940	26·80410	84·82000	3
1	·18768	·36875	·83461	2·37590	7·45665	23·62542	74·79030	2
0	·12844	·27707	·68853	2·03404	6·43623	20·43920	64·74942	1
$\nu'' = 0$	·050126	·15938	·51226	1·66997	5·40117	17·23952	54·68967	0
$\nu = 6$	−·35914	−·63954	−1·24438	−3·39745	−10·52917	−33·19477	−104·9055	$n-s = 5$
5	−·33085	−·59225	−1·15102	−3·08781	−9·53358	−30·03375	−94·89800	4
4	−·29968	−·53971	−1·05363	−2·77866	−8·53916	−26·87357	−84·88943	3
3	−·26454	−·48007	−·94873	−2·46938	−7·54638	−23·71475	−74·87961	2
2	−·22346	−·40993	−·83059	−2·15805	−6·55582	−20·55830	−64·86847	1
1	−·17165	−·32110	−·68842	−1·83919	−5·56813	−17·40621	−54·85633	0
0	−·078057	−·11507	−·14898	−·16391	−·16637	−·16664	−·16666	$n'-s = 0$
$\nu' = 1$	−·015521	−·042244	−·087229	−·11391	−·11850	−·11899	−·11904	1
2	−·0095802	−·027532	−·062725	−·065126	−·088846	−·089241	−·089281	2
3	−·0069485	−·020563	−·049022	−·066419	−·069127	−·069413	−·069441	3
4	−·0054587	−·016472	−·040113	−·053390	−·055330	−·055533	−·055553	4
5	−·0044992	−·013779	−·033771	−·043894	−·045293	−·045438	−·045453	5

TABLE 6. EIGENVALUES λ WHEN $\epsilon < 0$ AND $s = 0$, FOR GIVEN VALUES OF η

$\log_{\sqrt{2}}\lvert\eta\rvert$	$\nu = 0$	0	1	1	2	2	3	3
9	± ·00543737	± ·00352334	± ·00124769	± ·000987967	± ·000536440	± ·000457407	± ·00029648	± ·0002627
8	·00768885	·00498264	·00176449	·00139719	·000758640	·000646871	·00041929	·0003715
7	·0108715	·00704618	·00249534	·00197592	·00107288	·000914812	·00059297	·0005254
6	·0153687	·00996387	·00352887	·00279435	·00151727	·00129374	·00083858	·0007431
5	·0217176	·0140884	·00499034	·00395174	·00214572	·00182961	·0011859	·001051
4	·0306656	·0199165	·00705676	·00558842	·00303446	·00258744	·0016772	·001486
3	·0432340	·0281451	·00997794	·00790267	·00429122	·00365914	·0023718	·002102
2	·0607715	·0397437	·0141058	·0111745	·00606826	·00517464	·0033542	·002972
1	·0849361	·0560396	·0199339	·0157989	·00858058	·00731758	·0047433	·004203
0	·117473	·0787894	·0281497	·0223307	·0121313	·0103473	·0067074	·005944
−1	·159599	·110163	·0396945	·0315460	·0171464	·0146296	·00948402	·00840517
−2	·211029	·152460	·0558169	·0445161	·0242209	·0206788	·0134075	·0118846
−3	·269383	·207288	·0780658	·0626853	·0341758	·0292145	·0189473	·0168014
−4	·331056	·274155	·108107	·0879073	·0481150	·0412321	·0267568	·0237439
−5	·393143	·349361	·147191	·122323	·0674495	·0580780	·0377311	·0335314
−6	·454472	·426846	·195311	·167862	·0937979	·0814919	·0530583	·0472875
−7	·514866	·500892	·250788	·225166	·128614	·113515	·0742141	·0665049
−8	·573734	·568207	·311281	·292316	·172484	·156061	·102794	·0930419
−9	·629591	·627933	·375234	·364761	·224627	·209957	·140044	·128907
−10	·680837	·680474	·441403	·437318	·283551	·273736	·186228	·175609
−11	·726568	·726513	·507474	·506404	·347991	·343504	·240628	·233043
−12	·766657	·766652	·570536	·570358	·415989	·414698	·302295	·298695
−13	·801433	·801432	·628517	·628500	·484188	·483972	·369518	·368504
−14	·831395	·831395	·680537	·680536	·549317	·549297	·438892	·438740
−15	·857086	·857086	·726484	·726284	·609382	·609381	·506685	·506674
−16	·879032	·879032	·766628	·766628	·663516	·663516	·570326	·570325
−17	·897724	·897724	·801419	·801419	·711526	·711526	·628450	·628450
−18	·913609	·913609	·831389	·831389	·753604	·753604	·680509	·680509
−19	·927082	·927082	·857083	·857083	·790161	·790161	·726471	·726471
−20	·938492	·938492	·879030	·879030	·821709	·821709	·766621	·766621
−21	·948144	·948144	·897724	·897724	·848796	·848796	·801416	·801416
−22	·956300	·956300	·913608	·913608	·871960	·871960	·83139	·83139
−23	·963186	·963186	·927082	·927082	·89171	·89171	—	—
−24	·968996	·968996	·938492	·938492	—	—	—	—

Table 7. Eigenvalues λ when $\epsilon < 0$ and $s = 1, 2, \ldots, 5$.

Modes travelling eastwards

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	0	1	1	2	2	3	3
				$s = 1$				
−1	·00749421	·00749421	·00200622	·00200622	·000903727	·000903727	—	—
−2	·0297842	·0297744	·00860000	·00860000	·00393970	·00393970	·00224041	·00224041
−3	·0663914	·0659819	·0207592	·0207541	·00970997	·00970989	·00556592	·00556592
−4	·116858	·114499	·0396389	·0395012	·0189891	·0189806	·0109940	·0109934
−5	·179618	·173904	·0666980	·0658100	·0327858	·0326644	·0191980	·0191792
−6	·250897	·242764	·103518	·100766	·0524491	·0518132	·0310916	·0309363
−7	·325891	·318218	·150998	·145690	·0796397	·0777601	·0479177	·0472847
−8	·400756	·395742	·208376	·201426	·115970	·112222	·0712454	·0695917
−9	·473068	·470779	·273189	·266954	·162274	·156928	·102756	·0996427
−10	·541087	·540367	·342466	·338732	·217926	·212549	·143707	·139351
−11	·603430	·603279	·413465	·412032	·281087	·277488	·194268	·189894
−12	·659301	·659281	·483362	·483027	·349408	·347925	·253427	·250531
−13	·708567	·708565	·549520	·549475	·419923	·419577	·319364	·318232
−14	·751530	·751530	·610204	·610200	·489239	·489197	·389210	·388977
−15	·788706	·788706	·664666	·664666	·554638	·554636	·459296	·459273
−16	·820688	·820688	·712812	·712812	·614565	·614565	·526486	·526485
−17	·848079	·848079	·754907	·754907	·668357	·668357	·588805	·588805
−18	·871456	·871456	·791409	·791409	·715928	·715928	·645254	·645254
−19	·891352	·891352	·822865	·822865	·757534	·757534	·695508	·695508
−20	·908250	·908250	·849842	·849842	·793622	·793622	·739681	·739681
−21	·922575	·922575	·872891	·872891	·824728	·824728	·778141	·778141
−22	·934702	·934702	·892526	·892526	·851410	·851410	·81139	·81139
−23	·944956	·944956	·909213	·909213	·87421	·87421	—	—
−24	·953619	·953619	·92337	·92337	—	—	—	—
				$s = 2$				
−3	·00681816	·00681816	·00256432	·00256432	·00132482	·00132482	—	—
−4	·0279210	·0279209	·0110048	·0110048	·00576745	·00576745	·00352863	·00352863
−5	·0641378	·0641102	·0266550	·0266544	·0142100	·0142099	·00875826	·00875826
−6	·115501	·115197	·0510630	·0510305	·0277981	·0277953	·0172920	·0172918
−7	·180314	·179343	·0858198	·0855372	·0479552	·0479001	·0301788	·0301683
−8	·254614	·252145	·132058	·131101	·0763457	·0760129	·0487652	·0486628
−9	·333255	·331995	·189506	·187798	·114572	·113587	·0746887	·0742474
−10	·411649	·410996	·255953	·254179	·163445	·161737	·109628	·108539
−11	·486472	·486265	·327924	·326839	·222240	·220442	·154693	·153002
−12	·555567	·555528	·401722	·401341	·288732	·287621	·209739	·208108
−13	·617806	·617802	·473938	·473865	·359792	·359415	·273227	·277318
−14	·672888	·672888	·541912	·541905	·431832	·431768	·342457	·342195
−15	·721038	·721038	·604054	·604054	·501538	·501533	·413852	·413818
−16	·762759	·762759	·659704	·659704	·566600	·566599	·483866	·483864
−17	·798675	·798675	·708808	·708808	·625774	·625774	·549902	·549902
−18	·829444	·829444	·751672	·751672	·678596	·678596	·610443	·610443
−19	·855706	·855706	·788790	·788790	·725109	·725109	·664809	·664809
−20	·878056	·878056	·820737	·820737	·765654	·765654	·712896	·712896
−21	·897034	·897034	·848108	·848108	·800730	·800730	·754956	·754956
−22	·913121	·913121	·871473	·871473	·830901	·830901	—	—
−23	·926737	·926737	·891362	·891362	—	—	—	—
−24	·938248	·938248	·90826	·90826	—	—	—	—

TABLE 7 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	0	1	1	2	2	3	3
				$s = 3$				
−4	·00396198	·00396198	·00181350	·00181350	—	—	—	—
−5	·0198851	·0198851	·00942807	·00942807	·00543198	·00543198	·00351618	·00351618
−6	·0492706	·0492699	·0242541	·0242541	·0141821	·0141821	·00924621	·00924621
−7	·0932565	·0932331	·0479442	·0479412	·0285294	·0285291	·0187591	·0187590
−8	·151730	·151596	·0822441	·0821960	·0500313	·0500204	·0332518	·0332495
−9	·222277	·221996	·128448	·128226	·0804450	·0803517	·0542300	·0541975
−10	·300270	·299985	·186479	·186021	·121349	·121024	·0834297	·0832590
−11	·380396	·380242	·254209	·253722	·173376	·172792	·122472	·122021
−12	·458205	·458160	·327877	·327604	·235466	·234900	·172164	·171499
−13	·530715	·530708	·403240	·403162	·304877	·304590	·231828	·231287
−14	·596346	·596345	·476557	·476546	·377866	·377795	·299186	·298963
−15	·654572	·654572	·545168	·545167	·450577	·450570	·370772	·370731
−16	·705519	·705519	·607600	·607600	·519922	·519921	·442746	·442743
−17	·749669	·749669	·663292	·663292	·583950	·583950	·511914	·511914
−18	·787662	·787662	·712273	·712273	·641710	·641710	·576177	·576177
−19	·820192	·820192	·754910	·754910	·692944	·692944	·634431	·634431
−20	·847940	·847940	·791746	·791746	·737837	·737837	·686299	·686299
−21	·871539	·871539	·823391	·823391	·776820	·776820	·731881	·731881
−22	·891566	·891566	·850458	·850458	·810443	·810443	—	—
−23	·908532	·908532	·873534	·873534	—	—	—	—
−24	·922886	·922886	—	—	—	—	—	—
				$s = 4$				
−5	·00493314	·00493314	·00259595	·00259595	—	—	—	—
−6	·0204630	·0204630	·0110880	·0110880	·00688613	·00688613	·00467318	·00467318
−7	·0480602	·0480602	·0268308	·0268308	·0168952	·0168952	·0115509	·0115509
−8	·0892687	·0892648	·0515412	·0515393	·0329711	·0329710	·0227327	·0227327
−9	·144732	·144705	·0870549	·870407	·0567958	·0567913	·0395710	·0395698
−10	·212974	·212912	·134723	·134656	·0902051	·0901665	·0637306	·0637135
−11	·289979	·289915	·194425	·194289	·134688	·134560	·0970181	·0969328
−12	·370362	·370330	·263856	·263728	·190533	·190329	·140891	·140688
−13	·449177	·449169	·338941	·338883	·256126	·255966	·195661	·195415
−14	·522976	·522976	·415130	·415117	·328099	·328041	·259886	·259739
−15	·589904	·589904	·488604	·488603	·402287	·402277	·330463	·330424
−16	·649298	·649298	·556844	·556844	·474862	·474862	·403419	·403415
−17	·701241	·701241	·618552	·618552	·543083	·543083	·475028	·475028
−18	·746211	·746211	·673319	·673319	·605380	·605380	·542570	·542570
−19	·784869	·784869	·721287	·721287	·661101	·661101	·604439	·604439
−20	·817933	·817933	·762903	·762903	·710206	·710206	·659927	·659927
−21	·846107	·846107	·798759	·798759	·753018	·753018	·708938	·708938
−22	·870048	·870048	·829493	·829493	·790049	·790049	—	—
−23	·890350	·890350	·855734	·855734	—	—	—	—
−24	·907536	·907536	—	—	—	—	—	—

Table 7 (*cont.*)

$\log_{\sqrt{2}}(-\eta)$	$\nu = 0$	0	1	1	2	2	3	3
				$s = 5$				
−6	·00796545	·00796545	·00464987	·00464987	·0030296	·0030296	—	—
−7	·0250566	·0250566	·0149946	·0149946	·00989458	·00989458	·00699035	·00699035
−8	·0534028	·0534028	·0327622	·0327622	·0219030	·0219030	·0155917	·0155917
−9	·0947276	·0947265	·0597512	·0597507	·0405371	·0405370	·0291028	·0291028
−10	·149942	·149935	·0978447	·0978386	·0676031	·0676002	·0490465	·0490455
−11	·217882	·217867	·148298	·148273	·104962	·104942	·0772306	·0772188
−12	·294745	·294731	·210687	·210646	·153874	·153820	·115448	·115400
−13	·375137	·375132	·282273	·282242	·214065	·213998	·164808	·164720
−14	·453981	·453980	·358602	·358592	·283162	·283125	·224879	·224802
−15	·527724	·527724	·435002	·435001	·357178	·357170	·293279	·293249
−16	·594480	·594480	·507817	·507817	·431768	·431768	·366171	·366167
−17	·653603	·653603	·574804	·574804	·503383	·503383	·439439	·439439
−18	·705205	·705205	·634932	·634932	·569730	·569730	·509740	·509740
−19	·749798	·749798	·687987	·687987	·629650	·629650	·574903	·574903
−20	·788070	·788070	·734245	·734245	·682802	·682802	·633820	·633820
−21	·820758	·820758	·774234	·774234	·729346	·729346	·68615	·68615
−22	·848578	·848578	·808589	·808589	·76973	·76973	—	—
−23	·872195	·872195	·83797	·83797	—	—	—	—
−24	·892203	·892203	—	—	—	—	—	—

Table 8. Eigenvalues λ when $\epsilon < 0$ and $s = 1, 2, \ldots, 5$.

Modes travelling westwards

$\log_{\sqrt{2}} \eta$	$n-s = 0$	1	2	3	4	5	6	7
				$s = 1$				
9	−·501106	−·170759	−·0853249	−·0511643	−·0340971	−·0243495	−·0182594	−·0142003
8	−·501565	−·172471	−·0861508	−·0516463	−·0344129	−·0245726	−·0184256	−·0143289
7	−·502215	−·174908	−·0873195	−·0523276	−·0348590	−·0248878	−·0186602	−·0145105
6	−·503135	−·178388	−·0889736	−·0532905	−·0354890	−·0253325	−·0189912	−·0147665
5	−·504439	−·183372	−·0913145	−·0546514	−·0363783	−·0259598	−·0194578	−·0151274
4	−·506288	−·190541	−·0946261	−·0565746	−·0376328	−·0268439	−·0201150	−·0156355
3	−·508912	−·200905	−·0993062	−·0592928	−·0394016	−·0280889	−·0210397	−·0163499
2	−·512641	−·215962	−·105905	−·0631376	−·0418943	−·0298407	−·0223393	−·0173532
1	−·517942	−·237894	−·115164	−·0685862	−·0454059	−·0323048	−·0241642	−·0187605
0	−·525478	−·269679	−·128043	−·0763361	−·0503512	−·0357725	−·0267260	−·0207335
−1	−·536168	−·314659	−·145681	−·0874242	−·0573096	−·0406625	−·0303230	−·0235009
−2	−·551218	−·374651	−·169220	−·103398	−·0670761	−·0475885	−·0353772	−·0273903
−3	−·572027	−·446744	−·199393	−·126487	−·0807006	−·0574647	−·0424813	−·0328796
−4	−·599722	−·522817	−·235976	−·159553	−·0994729	−·0716484	−·0524505	−·0406779
−5	−·634551	−·594406	−·277672	−·205293	−·124759	−·0920636	−·0663641	−·0518360
−6	−·674572	−·657059	−·323173	−·264363	−·157563	−·121131	−·0855549	−·0678567
−7	−·716452	−·710171	−·372622	−·333653	−·197884	−·161218	−·111448	−·0907143
−8	−·756741	−·754931	−·427218	−·407104	−·244678	−·213462	−·145119	−·122629
−9	−·793223	−·792817	−·486678	−·478908	−·297262	−·276450	−·186704	−·165450
−10	−·825082	−·825014	−·547930	−·545767	−·356120	−·346044	−·235607	−·219621
−11	−·852389	−·852381	−·607012	−·606597	−·420553	−·417238	−·291806	−·283192
−12	−·875593	−·875593	−·661274	−·661222	−·487076	−·486381	−·355172	−·352153
−13	−·895229	−·895229	−·709684	−·709680	−·551564	−·551478	−·423003	−·422375
−14	−·911809	−·911809	−·752170	−·752170	−·611363	−·611357	−·490960	−·490890
−15	−·925790	−·925790	−·789075	−·789075	−·665330	−·665329	−·555623	−·555620
−16	−·937567	−·937567	−·820901	−·820901	−·713193	−·713193	−·615130	−·615130
−17	−·947483	−·947483	−·848203	−·848203	−·755126	−·755126	−·668682	−·668682
−18	−·955829	−·955829	−·871528	−·871528	−·791536	−·791536	−·716115	−·716115
−19	−·962851	−·962851	−·891394	−·891394	−·822939	−·822939	−·757642	−·757642
−20	−·968758	−·968758	−·908274	−·908274	−·849885	−·849885	−·793685	−·793685
−21	−·973727	−·973727	−·922589	−·922589	−·872916	−·872916	−·824764	−·824764
−22	−·977906	−·977906	−·934710	−·934710	−·892541	−·892541	−·851431	−·851431
−23	−·981420	−·981420	−·944961	−·944961	−·909222	−·909222	−·87422	−·87422
−24	−·984376	−·984376	−·953622	−·953622	−·92337	−·92337	—	—

Table 8 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=2$				
9	−·333802	−·167999	−·100958	−·0673456	−·0481178	−·0360941	−·0280759	−·0224622
8	−·333997	−·168553	−·101356	−·0676268	−·0483242	−·0362513	−·0281993	−·0225615
7	−·334273	−·169338	−·101918	−·0680245	−·0486161	−·0364734	−·0283737	−·0227018
6	−·334664	−·170451	−·102713	−·0685868	−·0490286	−·0367873	−·0286200	−·0229000
5	−·335221	−·172031	−·103840	−·0693821	−·0496116	−·0372308	−·0289678	−·0231797
4	−·336012	−·174278	−·105435	−·0705065	−·0504352	−·0378568	−·0294586	−·0235745
3	−·337141	−·177479	−·107693	−·0720959	−·0515981	−·0387402	−·0301509	−·0241310
2	−·338757	−·182049	−·110889	−·0743418	−·0532392	−·0399856	−·0311263	−·0249148
1	−·341081	−·188592	−·115408	−·0775135	−·0555529	−·0417397	−·0324990	−·0260172
0	−·344447	−·197981	−·121780	−·0819888	−·0588116	−·0442073	−·0344283	−·0275655
−1	−·349362	−·211467	−·130710	−·0882976	−·0633944	−·0476743	−·0371362	−·0297368
−2	−·356616	−·230781	−·143079	−·0971847	−·0698263	−·0525401	−·0409319	−·0327782
−3	−·367451	−·258138	−·159852	−·109703	−·0788236	−·0593659	−·0462470	−·0370351
−4	−·383774	−·295881	−·181809	−·127343	−·0913285	−·0689483	−·0536815	−·0429950
−5	−·408235	−·345411	−·209095	−·152132	−·108485	−·0824285	−·0640554	−·0513536
−6	−·443552	−·405539	−·241086	−·186519	−·131453	−·101426	−·0784394	−·0631141
−7	−·490460	−·471865	−·277518	−·232602	−·160953	−·128092	−·0980997	−·0797116
−8	−·545828	−·538726	−·320206	−·290512	−·196788	−·164821	−·124238	−·103090
−9	−·603907	−·601850	−·371972	−·357133	−·238567	−·213250	−·157467	−·135530
−10	−·659555	−·659118	−·432577	−·427242	−·287657	−·272731	−·197653	−·178944
−11	−·709902	−·709837	−·497560	−·496246	−·345883	−·339782	−·245343	−·233602
−12	−·754026	−·754020	−·561581	−·561372	−·411405	−·409796	−·302164	−·297244
−13	−·792051	−·792051	−·621153	−·621134	−·479252	−·478998	−·367170	−·365909
−14	−·824520	−·824520	−·674776	−·674775	−·544904	−·544882	−·435961	−·435783
−15	−·852093	−·852093	−·722111	−·722111	−·605755	−·605754	−·503898	−·503886
−16	−·875429	−·875429	−·763372	−·763372	−·660674	−·660674	−·567952	−·567952
−17	−·895137	−·895137	−·799027	−·799027	−·709363	−·709363	−·626548	−·626548
−18	−·911756	−·911756	−·829648	−·829648	−·751991	−·751991	−·679040	−·679040
−19	−·925759	−·925759	−·855824	−·855824	−·788974	−·788974	−·725364	−·725364
−20	−·937549	−·937549	−·878124	−·878124	−·820844	−·820844	−·765801	−·765801
−21	−·947473	−·947473	−·897074	−·897074	−·848170	−·848170	−·800815	−·800815
−22	−·955823	−·955823	−·913144	−·913144	−·871509	−·871509	−·830950	−·830950
−23	−·962847	−·962847	−·926750	−·926750	−·891383	−·891383	—	—
−24	−·968756	−·968756	−·938256	−·938256	−·90827	−·90827	—	—

Table 8 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=3$				
9	−·250231	−·150613	−·100536	−·0718577	−·0539132	−·0419423	−·0335591	−·0274605
8	−·250326	−·150868	−·100759	−·0720355	−·0540547	−·0420564	−·0336526	−·0275382
7	−·250462	−·151228	−·101073	−·0722870	−·0542549	−·0422178	−·0337847	−·0276481
6	−·250655	−·151739	−·101519	−·0726428	−·0545380	−·0424459	−·0339715	−·0278034
5	−·250928	−·152463	−·102150	−·0731462	−·0549383	−·0427683	−·0342355	−·0280228
4	−·251317	−·153492	−·103043	−·0738585	−·0555043	−·0432241	−·0346084	−·0283328
3	−·251872	−·154954	−·104310	−·0748665	−·0563044	−·0438680	−·0351350	−·0287703
2	−·252666	−·157038	−·106106	−·0762928	−·0574354	−·0447773	−·0358783	−·0293876
1	−·253807	−·160016	−·108653	−·0783110	−·0590331	−·0460606	−·0369266	−·0302576
0	−·255460	−·164283	−·112265	−·0811653	−·0612886	−·0478700	−·0384033	−·0314825
−1	−·257876	−·170421	−·117380	−·0851976	−·0644689	−·0504180	−·0404808	−·0332044
−2	−·261455	−·179279	−·124587	−·0908836	−·0689446	−·0539999	−·0433984	−·0356208
−3	·266858	−·192081	−·134642	−·0988802	−·0752258	−·0590248	−·0474879	−·0390056
−4	−·275204	−·210524	−·148383	−·110092	−·0840006	−·0660584	−·0532080	−·0437388
−5	−·288426	−·236760	−·166485	−·125766	−·0961591	−·0758842	−·0611884	−·0503490
−6	−·309646	−·273052	−·189040	−·147600	−·112741	−·0895941	−·0722745	−·0595753
−7	−·342767	−·320825	−·215653	−·177733	−·134681	−·108703	−·0875334	−·0724591
−8	−·389984	−·379315	−·247254	−·218315	−·162298	−·135208	−·108121	−·0904576
−9	−·448996	−·444972	−·287627	−·270381	−·195310	−·171344	−·134875	−·115506
−10	−·513872	−·512756	−·340121	−·332478	−·234952	−·218712	−·167913	−·149807
−11	−·578425	−·578211	−·403052	−·400663	−·284745	−·276913	−·207823	−·195056
−12	−·638504	−·638477	−·470789	−·470300	−·345516	−·342980	−·257337	−·251177
−13	−·692200	−·692198	−·537717	−·537657	−·413180	−·412675	−·317590	−·315699
−14	−·739073	−·739073	−·600264	−·600260	−·482179	−·482124	−·385132	−·384800
−15	−·779425	−·779425	−·656774	−·656774	−·548358	−·548355	−·454754	−·454725
−16	−·813870	−·813870	−·706763	−·706763	−·609409	−·609409	−·522318	−·522317
−17	−·843118	−·843118	−·750375	−·750375	−·664320	−·664320	−·585319	−·585319
−18	−·867872	−·867872	−·788066	−·788066	−·712859	−·712859	−·642491	−·642491
−19	−·888775	−·888775	−·820425	−·820425	−·755246	−·755246	−·693390	−·693390
−20	−·906403	−·906403	−·848074	−·848074	−·791940	−·791940	−·738093	−·738093
−21	−·921256	−·921256	−·871617	−·871617	−·823502	−·823502	−·776968	−·776968
−22	−·933761	−·933761	−·891611	−·891611	−·850523	−·850523	−·81053	−·81053
−23	−·944287	−·944287	−·908559	−·908559	−·873572	−·873572	—	—
−24	−·953143	−·953143	−·922901	−·922901	—	—	—	—

TABLE 8 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s = 4$				
9	−·200129	−·133670	−·0955695	−·0717169	−·0557994	−·0446502	−·0365381	−·0304522
8	−·200182	−·133809	−·0957069	−·0718363	−·0559005	−·0447354	−·0366104	−·0305140
7	−·200258	−·134007	−·0959013	−·0720054	−·0560434	−·0448559	−·0367126	−·0306014
6	−·200366	−·134287	−·0961766	−·0722445	−·0562455	−·0450263	−·0368571	−·0307249
5	−·200518	−·134684	−·0965665	−·0725830	−·0565314	−·0452673	−·0370614	−·0308995
4	−·200735	−·135247	−·0971188	−·0730621	−·0569360	−·0456081	−·0373502	−·0311462
3	−·201044	−·136048	−·0979020	−·0737406	−·0575083	−·0460900	−·0377583	−·0314949
2	−·201485	−·137188	−·0990134	−·0747016	−·0583181	−·0467713	−·0383350	−·0319872
1	−·202118	−·138815	−·100592	−·0760634	−·0594637	−·0477342	−·0391495	−·0326823
0	−·203033	−·141146	−·102837	−·0779933	−·0610843	−·0490944	−·0402991	−·0336625
−1	−·204366	−·144499	−·106030	−·0807281	−·0633754	−·0510145	−·0419200	−·0350436
−2	−·206335	−·149350	−·110568	−·0845996	−·0666104	−·0537212	−·0442022	−·0369863
−3	−·209294	−·156407	−·116995	−·0900687	−·0711685	−·0575290	−·0474091	−·0397138
−4	−·213864	−·166721	−·125997	−·0977657	−·0775674	−·0628714	−·0519045	−·0435345
−5	−·221171	−·181807	−·138319	−·108547	−·0864961	−·0703405	−·0581866	−·0488735
−6	−·233328	−·203704	−·154449	−·123580	−·0988177	−·0807440	−·0669298	−·0563158
−7	−·253959	−·234783	−·174250	−·144462	−·115454	−·0951880	−·0790148	−·0666683
−8	−·287436	−·277093	−·197799	−·173252	−·136995	−·115188	−·0954791	−·0810492
−9	−·335738	−·331165	−·228096	−·212118	−·163304	−·142716	−·117235	−·100995
−10	−·396487	−·394905	−·270301	−·262351	−·195060	−·179922	−·144563	−·128485
−11	−·464217	−·463824	−·326028	−·323075	−·236305	−·228249	−·177848	−·165631
−12	−·532864	−·532800	−·391668	−·390910	−·290214	−·287193	−·220229	−·213782
−13	−·597839	−·597833	−·461351	−·461231	−·354611	−·353872	−·274638	−·272370
−14	−·656649	−·656649	−·529830	−·529820	−·424199	−·424096	−·339122	−·338639
−15	−·708424	−·708424	−·593742	−·593741	−·493784	−·493777	−·408731	−·408678
−16	−·753235	−·753235	−·651449	−·651449	−·559779	−·559779	−·478582	−·478579
−17	−·791617	−·791617	−·702452	−·702452	−·620211	−·620211	−·545208	−·545208
−18	−·824278	−·824278	−·746896	−·746896	−·674259	−·674259	−·606589	−·606589
−19	−·851957	−·851957	−·785259	−·785259	−·721822	−·721822	−·661789	−·661789
−20	−·875352	−·875352	−·818156	−·818156	−·763209	−·763209	−·710599	−·710599
−21	−·895092	−·895092	−·846236	−·846236	−·798935	−·798935	−·753243	−·753243
22	−·911731	−·911731	−·870123	−·870123	−·829595	−·829595	−·79018	−·79018
−23	−·925744	−·925744	−·890393	−·890393	−·85579	−·85579	—	—
24	−·937541	−·937541	−·907562	−·907562	—	—	—	—

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE 8 (*cont.*)

$\log_{\sqrt{2}} \eta$	$n-s=0$	1	2	3	4	5	6	7
				$s=5$				
9	−·166745	−·119253	−·0895051	−·0696474	−·0557355	−·0456119	−·0380162	−·0321716
8	−·166778	−·119338	−·0895961	−·0697316	−·0558100	−·0456771	−·0380731	−·0322214
7	−·166825	−·119459	−·0897249	−·0698506	−·0559155	−·0457694	−·0381536	−·0322919
6	−·166890	−·119630	−·0899072	−·0700190	−·0560647	−·0458998	−·0382675	−·0323915
5	−·166983	−·119873	−·0901653	−·0702575	−·0562758	−·0460843	−·0384285	−·0325323
4	−·167116	−·120217	−·0905312	−·0705951	−·0565745	−·0463453	−·0386561	−·0327315
3	−·167304	−·120705	−·0910499	−·0710733	−·0569973	−·0467146	−·0389781	−·0330131
2	−·167573	−·121400	−·0917863	−·0717510	−·0575960	−·0472370	−·0394333	−·0334110
1	−·167959	−·122392	−·0928329	−·0727122	−·0584438	−·0479760	−·0400770	−·0339733
0	−·168514	−·123812	−·0943227	−·0740764	−·0596447	−·0490216	−·0409866	−·0347675
−1	−·169320	−·125854	−·0964469	−·0760135	−·0613459	−·0505003	−·0422715	−·0358883
−2	−·170505	−·128808	−·0994795	−·0787648	−·0637547	−·0525898	−·0440847	−·0374682
−3	−·172277	−·133113	−·103808	−·0826691	−·0671613	−·0555386	−·0466396	−·0396919
−4	−·174992	−·139442	−·109961	−·0881952	−·0719664	−·0596897	−·0502312	−·0428146
−5	−·179313	−·148819	−·118590	−·0959796	−·0787115	−·0655124	−·0552642	−·0471875
−6	−·186541	−·162757	−·130319	−·106879	−·0881007	−·0736410	−·0622875	−·0532912
−7	−·199280	−·183332	−·145326	−·122068	−·100960	−·0849291	−·0720287	−·0617803
−8	−·221795	−·212995	−·163348	−·143191	−·117987	−·100539	−·0853890	−·0735474
−9	−·257999	−·253949	−·186116	−·172397	−·139227	−·122069	−·103269	−·0898179
−10	−·308610	−·307041	−·219001	−·211912	−·164826	−·151577	−·126066	−·112258
−11	−·371102	−·370628	−·265854	−·262987	−·198522	−·191179	−·153921	−·142934
−12	−·440569	−·440470	−·325542	−·324674	−·245029	−·242053	−·189875	−·183832
−13	−·511280	−·511268	−·393669	−·393497	−·304132	−·303282	−·238139	−·235830
−14	−·578604	−·578603	−·464671	−·464651	−·371781	−·371634	−·298286	−·297711
−15	−·639840	−·639840	−·533754	−·533753	−·442671	−·442658	−·366285	−·366207
−16	−·693932	−·693932	−·597852	−·597852	−·512187	−·512187	−·437094	−·437090
−17	−·740852	−·740852	−·655489	−·655489	−·577272	−·577272	−·506438	−·506438
−18	−·781094	−·781094	−·706265	−·706265	−·636324	−·636324	−·571466	−·571466
−19	−·815368	−·815368	−·750397	−·750397	−·688774	−·688774	−·630635	−·630635
−20	−·844430	−·844430	−·788411	−·788411	−·734692	−·734692	−·683361	−·683361
−21	−·869003	−·869003	−·820953	−·820953	−·774489	−·774489	−·729666	−·729666
−22	−·889743	−·889743	−·848691	−·848691	−·808736	−·808736	−·76991	−·76991
−23	−·907226	−·907226	−·872260	−·872260	−·83806	−·83806	—	—
−24	−·921953	−·921953	−·89224	−·89224	—	—	—	—

Table 9. Values of $(-\epsilon)^{-\frac{1}{2}}$ interpolated at fixed values of λ when $s = 0, 1, 2$

λ	$\nu = 0$	0	1	1	2	2	3	3	4	4
					$s = 0$					
±0·9	·048718	·048718	·024359	·024359	·016239	·016239	·01218	·01218	·009694	·009694
0·8	·094739	·094738	·047366	·047366	·031577	·031577	·023683	·023683	·018946	·018946
0·7	·13790	·13781	·068905	·068905	·045934	·045934	·034450	·034450	·027560	·027560
0·6	·17876	·17707	·088852	·088836	·059219	·059218	·044411	·044411	·035528	·035528
0·5	·21945	·21068	·10722	·10687	·071334	·071310	·053483	·053481	·042782	·042782
0·4	·26085	·23751	·12455	·12230	·082312	·081951	·061601	·061528	·049249	·049232
0·3	·29867	·25763	·14131	·13439	·092515	·090609	·068887	·068240	·054934	·054686
0·2	·32747	·27153	·15569	·14288	·10172	·096832	·075480	·073175	·06000	·058780
0·1	·34496	·27968	·16492	·14787	·10802	·10052	·080247	·076126	·063813	·061252
0·05	·34933	·28169	·1672	·14911	·10964	·10143	·08149	·076857	·064829	·061866
0·02	·35055	·28225	·16790	·14945	·11009	·10169	·081840	·077061	·065114	·062037
0·01	·35072	·28233	·16799	·14950	·11015	·10172	·081890	·077090	·065155	·062062
					$s = 1$					
0·9	·03238	·03238	·01947	·01947	·01391	·01391	·01082	·01082	·008674	·008674
0·8	·06272	·06272	·03780	·03780	·02703	·02703	·02103	·02103	·01721	·01721
0·7	·09079	·09079	·05488	·05488	·03928	·03928	·03058	·03058	·02503	·02503
0·6	·1163	·1162	·07058	·07058	·05058	·05058	·03939	·03939	·03225	·03225
0·5	·1390	·1385	·08471	·08467	·06079	·06079	·04738	·04738	·03881	·03881
0·4	·1584	·1566	·09716	·09675	·06981	·06972	·05444	·05442	·04461	·04460
0·3	·1728	·1697	·1076	·1061	·07753	·07696	·06049	·06025	·04957	·04947
0·2	·1786	·1756	·1145	·1123	·08336	·08199	·06530	·06446	·05359	·05307
0·1	·1667	·1658	·1135	·1122	·08483	·08363	·06737	·06636	·05573	·05491
0·05	·1425	·1424	·1039	·1037	·08055	·08019	·06530	·06489	·05470	·05428
0·02	·1065	·1065	·08482	·08482	·06965	·06963	·05864	·05862	·05040	·05037
0·01	·08185	·08185	·06895	·06895	·05906	·05906	·05135	·05135	·04522	·04522
−0·9	·09515	·09515	·03239	·03239	·01947	·01947	·01391	·01391	·01082	·01082
−0·8	·1816	·18133	·06282	·06282	·03782	·03782	·02704	·02704	·02104	·02104
−0·7	·2663	·2577	·09121	·09121	·05497	·05497	·03932	·03932	·03060	·03060
−0·6	·3868	·3210	·1176	·1174	·07085	·07085	·05068	·05068	·03944	·03944
−0·5	—	·3725	·1431	·1410	·08543	·08533	·06104	·06104	·04750	·04750
−0·4	—	·4197	·1720	·1608	·09924	·09806	·07046	·07026	·05473	·05469
−0·3	—	·4845	·2112	·1767	·1142	·1085	·07953	·07799	·06132	·06080
−0·2	—	·7618	·2651	·1914	·1319	·1165	·08946	·08385	·06803	·06548
−0·1	—	—	·5207	·2309	·1575	·1260	·1009	·08886	·07512	·06896

TABLE 9 (*cont.*)

λ	$\nu = 0$	0	1	1	2	2	3	3	4	4
					$s = 2$					
0·9	·02420	·02420	·01619	·01619	·01216	·01216	·009718	·009718	·00767	·00767
0·8	·04668	·04668	·03137	·03137	·02360	·02360	·01890	·01890	·01576	·01576
0·7	·06728	·06728	·04544	·04544	·03424	·03424	·02745	·02745	·02290	·02290
0·6	·08574	·08573	·05826	·05826	·04400	·04400	·03532	·03532	·02948	·02948
0·5	·1017	·1017	·06966	·06965	·05276	·05276	·04241	·04241	·03543	·03543
0·4	·1148	·1146	·07938	·07931	·06037	·06034	·04862	·04861	·04066	·04066
0·3	·1238	·1235	·08699	·08671	·06657	·06642	·05377	·05369	·04505	·04501
0·2	·1268	·1264	·09135	·09091	·07077	·07039	·05753	·05724	·04836	·04816
0·1	·1172	·1172	·08872	·08855	·07063	·07041	·05837	·05812	·04959	·04934
0·05	—	—	—	—	·06552	·06549	·05536	·05532	·04776	·04770
−0·9	·04754	·04754	·02421	·02421	·01619	·01619	·01217	·01217	·009715	·009715
−0·8	·09034	·09034	·04677	·04677	·03140	·03140	·02361	·02361	·01891	·01891
−0·7	·1289	·1289	·06762	·06762	·04554	·04554	·03428	·03428	·02747	·02747
−0·6	·1648	·1637	·08675	·08674	·05857	·05857	·04413	·04413	·03538	·03538
−0·5	·2037	·1956	·1044	·1041	·07044	·07041	·05309	·05309	·04258	·04258
−0·4	·2799	·2270	·1224	·1196	·08141	·08099	·06118	·06110	·04902	·04900
−0·3	—	·2695	·1481	·1334	·09311	·09008	·06889	·06798	·05487	·05454
−0·2	—	·4339	·1985	·1495	·1102	·09819	·07833	·07365	·06120	·05901
−0·1	—	—	—	·2139	·1439	·1131	·09268	·08064	·06975	·06337

TABLE 10. LIMITING VALUES OF ϵ AS $\lambda \to 0$ WHEN $s = 0$ AND $\epsilon < 0$

m	$(-\epsilon)^{-\frac{1}{2}}$	$(-\epsilon)^{\frac{1}{2}}$	ϵ
1	·350779	2·85079	−8·12700
2	·282358	3·54160	−12·5429
3	·168025	5·95150	−35·4204
4	·149517	6·68820	−44·7320
5	·110174	9·07655	−82·3838
6	·101735	9·82945	−96·6181
7	·0819064	12·2091	−149·061
8	·0771000	12·9702	−168·225
9	·0652	15·3	−235
10	·0621	16·1	−260
11	·054	18	−340
12	·051	19	−370